Discrete Applied Mathematics 162 (2014) 195–201

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Algorithms for unipolar and generalized split graphs

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ARTICLE INFO

Article history:
Received 29 June 2011
Received in revised form 2 August 2013
Accepted 14 August 2013
Available online 1 October 2013

Keywords:
Split graph
Clique-split graph
Unipolar graph
Generalized split graph
Minimal triangulation
Perfect code
Efficient dominating set

ABSTRACT

A graph $G = (V, E)$ is a *unipolar graph* if there exists a partition $V = V_1 \cup V_2$ such that V_1 is a clique and V_2 induces the disjoint union of cliques. The complement-closed class of *generalized split graphs* contains those graphs G such that either G *or* the complement of G is unipolar. Generalized split graphs are a large subclass of perfect graphs. In fact, it has been shown that almost all C_5-free (and hence, almost all perfect graphs) are generalized split graphs. In this paper, we present a recognition algorithm for unipolar graphs that utilizes a minimal triangulation of the given graph, and produces a partition when one exists. Our algorithm has running time $O(nm + nm_F)$, where m_F is the number of edges added in a minimal triangulation of the given graph. Generalized split graphs can be recognized via this algorithm in $O(n^3)$ time. We give algorithms on unipolar graphs for finding a maximum independent set and a minimum clique cover in $O(n+m)$ time, and for finding a maximum clique and a minimum proper coloring in $O(n^{2.5}/\log n)$ time, when a unipolar partition is given. These algorithms yield algorithms for the four optimization problems on generalized split graphs that have the same worst-case time bounds. We also report that the perfect code problem is NP-complete for chordal unipolar graphs.

1. Introduction

The graphs in this paper are finite, simple, and undirected. For graph-theoretic terms not defined here and well-known graph theory concepts, see [9].

The class of polar graphs, introduced by Tyshkevich and Chernyak [46] in 1985, has received a lot of attention recently. A *complete multipartite graph* is the complement of a disjoint union of complete graphs. A graph G is *polar* if its vertex set can be partitioned into two sets A and B such that the subgraph induced by A in G is a complete multipartite graph and the subgraph induced by B in G is the complement of a complete multipartite graph (i.e., the disjoint union of complete graphs). When A is restricted to be an independent set, G is said to be *monopolar*; when A is restricted to be a clique, G is said to be *unipolar*. Unipolar graphs have also been called *clique-split* graphs in the literature [45]. A graph is a *split graph* if its vertex set can be partitioned into an independent set and a clique. The complement of a split graph is also a split graph. Polar graphs generalize bipartite and split graphs. If one replaces one of the bipartitions of a bipartite graph with a disjoint union of cliques, then a monopolar graph is obtained. If one replaces the independent set of a split graph with a disjoint union of cliques, then a unipolar graph is obtained.

If the complement of a graph G belongs to a given class $\mathcal{C}$, then we say that G is *co-*$\mathcal{C}$. For example, a graph whose complement is unipolar is said to be co-unipolar. The class of *generalized split graphs* is equal to the union of the class of unipolar graphs and the class of co-unipolar graphs. The graph classes unipolar, co-unipolar, and generalized split graphs are hereditary. The class of polar graphs is closed under complementation, while the complement of a monopolar (respectively,

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http://dx.doi.org/10.1016/j.dam.2013.08.011

unipolar) graph is polar but not necessarily monopolar (respectively, unipolar). Note that the class of generalized split graphs is closed under complementation.

The recognition problem for a graph class $\mathcal{C}$ is as follows: Given a graph G, is G a member of class $\mathcal{C}$? The recognition problem for both polar graphs [12] and monopolar graphs [24] has been shown to be NP-complete. Recent research is focused on the polar and monopolar recognition problems on special classes of graphs. Churchley and Huang [15] prove that testing for polarity remains NP-complete for claw-free graphs and testing for monopolarity remains NP-complete for triangle-free graphs, while showing that monopolarity can be decided efficiently for claw-free graphs. Le and Nevries [39,40] extend NP-completeness results for both recognition problems to several special classes of graphs, and establish that hole-free, P_5-free, $(2K_2, C_5)$-free, and chair-free graphs have polynomial-time monopolarity tests, while polarity is NP-complete for these classes. Polynomial-time algorithms for recognizing polar and monopolar graphs are known for a host of other special classes of graphs, including cographs [23], chordal graphs [21], permutation graphs [20], line graphs [14,22,34], and graph classes with bounded tree-width [3,17] or bounded clique-width [18].

Tyshkevich and Chernyak [47] showed that unipolar graphs can be recognized in $O(n^3)$ time. Churchley and Huang [16] give an alternate $O(n^2m)$-time algorithm using a reduction to a polynomial-time solvable 2-edge-colored homomorphism problem. In this paper, we present a recognition algorithm for unipolar graphs that utilizes a minimal triangulation of the given graph. Our algorithm has running time $O(nm+nm_F)$, where m_F is the number of edges added in a minimal triangulation of the given graph. Thus, our algorithm may be more efficient, and in all cases is not less efficient, than the $O(n^3)$ algorithm. Also, it is more efficient than the $O(n^2m)$ algorithm when m_F is sufficiently small. When our unipolar recognition algorithm is given a graph in the class, the algorithm produces a unipolar partition in $O(nm + nm_F)$ time. Generalized split graphs can be recognized in $O(n^3)$ time via the unipolar recognition algorithm of Tyshkevich and Chernyak [47] or via the unipolar recognition algorithm presented in this paper (note we may have to work on the complement of the given graph).

We denote the chordless cycle on k vertices by C_k. We shall call a chordless cycle on five or more vertices a *hole* and the complement of a chordless cycle on five or more vertices an *antihole*. Holes and antiholes are designated even or odd depending on whether they have an even or odd number of vertices.

Generalized split graphs can contain even holes, C_4, and even antiholes (since an even antihole can be partitioned into two cliques). It is not difficult to see that holes and odd antiholes are not unipolar graphs (see Lemmas 1 and 2). Thus, unipolar, co-unipolar, and generalized split graphs are perfect via the Strong Perfect Graph Theorem [13], which states that a graph is *perfect* if and only if it does not contain an odd hole or an odd antihole as an induced subgraph.

A graph is *chordal* if it does not contain an induced cycle on four or more vertices. Földes and Hammer [27] proved that the class of split graphs is equivalent to the class of graphs that are both chordal and co-chordal. The class of split graphs is properly contained in the intersection of unipolar and co-unipolar; a P_5, for instance, is not a split graph, but is both unipolar and co-unipolar. However, the class of generalized split graphs is incomparable to both the classes of chordal and co-chordal graphs. The graph G_c consisting of a triangle and a P_5 joined by the single edge between a vertex of the triangle and an endpoint of the P_5 is chordal but not generalized split. The complement of G_c is co-chordal but not generalized split. A graph is *weakly triangulated* if contains neither a hole nor an antihole. Weakly triangulated graphs are a well-known class of perfect graphs that generalize chordal graphs and co-chordal graphs. Along with the fact that generalized split graphs can contain even holes and even antiholes, G_c also establishes that the class of generalized split graphs is incomparable to the class of weakly triangulated graphs. It is easy to find graphs in the intersection of generalized split with chordal, co-chordal, and weakly chordal graphs. A superclass of the class of perfect graphs is the class of C_5-free graphs. A graph is C_5-free if it does not contain an induced cycle of length 5.

The class of generalized split graphs was introduced by Prömel and Steger [42] in their probabilistic study of perfect graphs. Let $GS(n)$ denote the set of all labeled generalized split graphs on n vertices, $P(n)$ denote the set of all labeled perfect graphs on n vertices, and $F(n)$ denote the set of all labeled C_5-free graphs on n vertices. Prömel and Steger prove the following theorem, which provides a structural characterization of *almost all* C_5-free graphs.

Theorem 1 (*Prömel and Steger [42]*)**.** *Almost all C_5-free graphs are generalized split graphs in the sense that* $|GS(n)|/|F(n)| \to 1$, *as* $n \to \infty$.

Since $GS(n) \subset P(n) \subset F(n)$, this theorem implies that almost all perfect graphs are generalized split graphs. A consequence of this theorem is that properties established for generalized split graphs are immediately properties of almost all C_5-free (and almost all perfect) graphs. Bacsó et al. [5] employ this technique to show that the clique hypergraphs of almost all perfect graphs are 3-colorable.

Szwarcfiter and Maffray [45] posed the problem of solving optimization problems on unipolar graphs and generalized split graphs. In this paper, we consider the unweighted (cardinality) versions of four classical optimization problems. We give $O(n + m)$-time algorithms to find a maximum independent set and a minimum clique cover in a unipolar graph when a unipolar partition is given. We also give $O(n^{2.5}/\log n)$-time algorithms to find a maximum clique and a minimum proper coloring in a unipolar graph when a unipolar partition is given. These algorithms yield algorithms for the four optimization problems on generalized split graphs that have the same worst-case time bounds. If a unipolar partition is not given as input, finding a unipolar partition of the input graph can dominate the running time. These four optimization problems are NP-complete for arbitrary graphs (and even for many special graph classes). However, both the unweighted and weighted versions are solvable in polynomial time on perfect graphs due to a result of Grötschel, Lovász, and Schrijver [31]. This result