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On the subgraph epimorphism problem

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ABSTRACT

In this paper we study the problem of deciding the existence of a subgraph epimorphism between two graphs. Our interest in this variant of graph matching problem stems from the study of model reductions in systems biology, where large systems of biochemical reactions can be naturally represented by bipartite digraphs of species and reactions. In this setting, model reduction can be formalized as the existence of a sequence of vertex deletion and merge operations that transforms a first reaction graph into a second graph. This problem is in turn equivalent to the existence of a subgraph (corresponding to delete operations) epimorphism (i.e. surjective homomorphism, corresponding to merge operations) from the first graph to the second. In this paper, we study theoretical properties of subgraph epimorphisms in general directed graphs. We first characterize subgraph epimorphisms (SEPI), subgraph isomorphisms (SISO) and graph epimorphisms (EPI) in terms of graph transformation operations. Then we study the graph distance measures induced by these transformations. We show that they define metrics on graphs and compare them. On the algorithmic side, we show that the SEPI existence problem is NP-complete by reduction of SAT and present a constraint satisfaction algorithm that has been successfully used to solve practical SEPI problems on a large benchmark of reaction graphs from systems biology.

1. Introduction

Our interest in subgraph epimorphisms stems from the study of model reductions in systems biology, where large systems of biochemical reactions can be naturally represented by bipartite digraphs of species and reactions [14,10]. In this setting, one can define a very general notion of model reduction as a particular form of graph transformation and use it to compare models in systems biology model repositories [8].

Let us consider, for example, the reduction of Michaelis–Menten in Fig. 1. The left-hand side graph is a detailed model composed of three reactions where an enzyme E binds in a reversible manner to a substrate S to form a complex ES and release a product P. The right-hand side graph reduces this system to a single reaction catalyzed by the enzyme.

The reduced graph can be obtained from the source graph by a sequence of delete and merge operations on species and reaction vertices. These transformations can typically be justified in chemistry by considering: (i) reaction deletions for slow reverse reactions, (ii) reaction mergings for reaction chains with a limiting reaction, (iii) molecular species deletions for species in excess and (iv) molecular mergings for quasi-steady state approximations.

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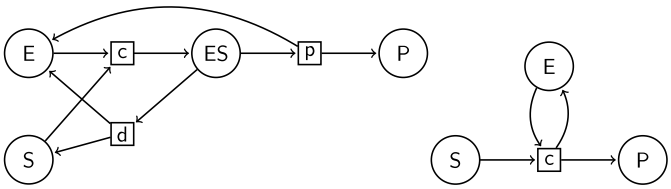

Fig. 1. A catalytic mechanism and the Michaelis–Menten reduced mechanism.

This operational view of graph reduction by graph transformation operations is equivalent to the existence of a subgraph (corresponding to delete operations) epimorphism (i.e. surjective homomorphism, corresponding to merge operations) from a source graph to a reduced graph. Subgraph epimorphisms (SEPI) differ from subgraph isomorphisms (SISO) by allowing merge operations in addition to delete operations. On undirected graphs, SEPIs differ from minors [13] with respect to the three following points: (i) non-adjacent vertices may be merged, (ii) merging adjacent vertices creates loops, and (iii) arcs cannot be deleted without deleting or merging vertices.

In this paper, we study the theoretical properties of SEPIs in general directed graphs and relate these properties with other standard notions of graph homomorphisms [9], namely subgraph isomorphisms, minors and graph epimorphisms (EPI).

Main results and overview of the paper. In Section 2, we introduce three partial orders on digraphs respectively based on SEPI , SISO and EPI , and show that, unlike the minor relation, they are not well quasi-orders. In Section 3, we introduce three graph distance measures, respectively based on SEPI , SISO and EPI , and we compare them. We show that they are metrics and that these distances are equivalent to graph edit distances defined as the minimum number of edit operations that transform a first graph into another one. In Section 5, we show the NP-completeness of the SEPI existence problem. In Section 6, we present a constraint satisfaction algorithm that has been successfully used to solve practical SEPI problems on a large benchmark of reaction graphs from systems biology. In Section 7, we discuss extensions to non -directed and bipartite graphs.

2. Partial order relations SISO , EPI and SEPI

2.1. Notations and definitions

A directed graph, or graph for short in this paper, is a pair (V, A) where V is a finite set of vertices and $A \subseteq V \times V$ a set of arcs. The cardinality of a set S is denoted as $|S|$. The size $|G|$ of a graph $G = (V, A)$ is its number of vertices, $|G| = |V|$.

For the remainder of this section, G and G' denote graphs, with $G = (V, A)$ and $G' = (V', A')$.

Definition 2.1 (*Graph Isomorphism*)**.** An isomorphism from G to G' is a bijective function $f : V \to V'$ such that $(u, v) \in A$ iff $(f(u), f(v)) \in A'$.

Two graphs G and G' are isomorphic when there exists a graph isomorphism from G to G'. Graph isomorphism is an equivalence relation on directed graphs: we note $\mathcal{G}$ the set of all graphs quotiented by this equivalence relation.

Definition 2.2 (*Graph Epimorphism*)**.** An epimorphism (EPI) from G to G' is a surjective function $f : V \to V'$ such that

- for all $u, v \in V$, if $(u, v) \in A$, then $(f(u), f(v)) \in A'$ (graph homomorphism), and,
- for all $(u', v') \in A'$, there exists $(u, v) \in A$ such that $f(u) = u'$ and $f(v) = v'$ (surjectivity on arcs).

If f is bijective, then f is a graph isomorphism. Graph epimorphisms relax the bijection constraint of graph isomorphisms to a surjection constraint on both vertices and arcs (hence the terminology of epimorphism) so that several vertices of G may be mapped on a same vertex of G'. Graph epimorphisms are closely related to graph compactions: on the class of irreflexive graphs (graphs without loops), graph epimorphisms are actually equivalent to graph compactions [19]. Graph epimorphisms are also closely related to quotient graphs (see Section 4.3).

Definition 2.3 (*Induced Subgraph*)**.** Let $U \subseteq V$ be a subset of vertices of G. The subgraph of G induced by U is $G_{\downarrow U} = (U, A \cap (U \times U))$.

Definition 2.4 (*Subgraph Isomorphism*)**.** A *subgraph isomorphism* (SISO) from G to G' is an isomorphism f from an induced subgraph G_0 of G to G'.

G_0 is the domain of f, denoted by $\operatorname{dom} f$.

Definition 2.5 (*Subgraph Epimorphism*)**.** A *subgraph epimorphism* (SEPI) from G to G' is an epimorphism f from an induced subgraph G_0 of G to G'.

G_0 is also denoted as $\operatorname{dom} f$.