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Characterization of facets of the hop constrained chain polytope via dynamic programming

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ABSTRACT

In this paper, we study the hop constrained chain polytope, that is, the convex hull of the incidence vectors of (s, t)-chains using at most k arcs of a given digraph, and its dominant. We use extended formulations (implied by the inherent structure of the Moore–Bellman–Ford algorithm) to derive facet defining inequalities for these polyhedra via projection. Our findings result in characterizations of all facet defining $0/\pm 1$-inequalities for the hop constrained chain polytope and all facet defining $0/1$-inequalities for its dominant. Although the derived inequalities are already known, such classifications were not previously given to the best of our knowledge. Moreover, we use this approach to generalize so called jump inequalities, which have been introduced in a paper by Dahl and Gouveia in 2004.

1. Introduction

Let $D = (V, A)$ be a directed graph without parallel arcs. An (s, t)-*chain* is a sequence of arcs $C = (a_1, a_2, \ldots, a_r)$ such that $a_i = (i_{p-1}, i_p)$ for $p = 1, \ldots, r$, with $i_0 = s$ and $i_r = t$. The nodes $i_1, i_2, \ldots, i_{r-1}$ are the *internal nodes* of C. If all arcs a_i are distinct, then C is called a *walk*; If all nodes i_p are distinct, then C is called a *path*. In what follows, chains will be usually denoted only as a sequence of nodes, but their incidence vectors are defined in the arc space $\mathbb{R}^A$. Here, for any chain C, its incidence vector $\chi^C \in \mathbb{R}^A$ is defined by

$$\chi^C_a := \text{number of times the arc } a \text{ is used by } C,$$

for $a \in A$. Note that different chains may have the same incidence vector.

Given a length function $d : A \to \mathbb{R}$, the *length* of a chain

$$C = (i_0, i_1, i_2, \ldots, i_q)$$

is defined as $d(C) := \sum_{p=1}^{q} d((i_{p-1}, i_p))$. In the *hop constrained shortest chain (walk, path) problem* we are looking for a chain (walk, path) using at most k arcs of minimum length. The hop constrained shortest path problem, which is known to be NP-hard, arises, for instance, in the design of telecommunication networks when data have to be sent along paths that must not contain more than a certain number of intermediate nodes in order to guarantee a minimum level of service quality [14,8].

The corresponding chain problem is a combinatorial relaxation of this problem which can be solved in polynomial time with the Moore–Bellman–Ford algorithm [3,11,24], see Algorithm 1. Using an integer programming approach for the hop constrained path problem, valid inequalities for the easier chain problem are of interest, since they are also valid inequalities

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for the harder problem. Thus, a branch-and-cut algorithm for solving the path problem, for example, directly benefits from efficient separation routines for the polyhedron associated with the chain problem.

Algorithm 1: Moore–Bellman–Ford

Input: A digraph $D = (V, A)$, a fixed node $s \in V$, and a length function $d : A \to \mathbb{R}$.

Output: For each node $j \in V$ and each number $\ell \in \{0, \dots, |V| - 1\}$, the length $u_j^{(\ell)}$ of a shortest (s, j)-chain using at most ℓ arcs and its predecessor $p(j, \ell)$ on such a chain. If j is not reachable from s, then $u_j^{(\ell)} = +\infty$ and $p(j, \ell)$ is undefined for all ℓ.

(1) Set $u_s^{(0)} := 0$ and $u_j^{(0)} := +\infty$ for all $j \in V \setminus \{s\}$.
(2) **for** $\ell := 1$ **to** $|V| - 1$ **do**
  Set $t_j := u_j^{(\ell-1)}$ for all $j \in V$.
  forall $(i, j) \in A$ **do**
   if $t_j > u_i^{(\ell-1)} + d((i, j))$ **then**
    Set $t_j := u_i^{(\ell-1)} + d((i, j))$ and $p(j, \ell) := i$.
  Set $u_j^{(\ell)} := t_j$ for all $j \in V$.

In this paper, we present some results on the *hop constrained chain polytope* $\mathcal{C}^{\le k}$, that is, the convex hull of the incidence vectors of chains using at most k arcs, and its *dominant* $\mathrm{dmt}(\mathcal{C}^{\le k}) := \mathcal{C}^{\le k} + \mathbb{R}_+^A$, where $\mathbb{R}_+^A$ is the nonnegative orthant. In the last years, closely related polyhedra have been investigated, see, for instance, [2,6,7,9,10,12,17–20,23,25,26], in particular the *hop constrained path polytope* $\mathcal{P}^{\le k}$ defined as the convex hull of the incidence vectors of hop constrained (s, t)-paths. Important for our context are the following three results.

Fact 1 *([26]). The integer points of $\mathcal{P}^{\le k}$ are characterized by the system*

$$x_{ii} = 0, \qquad (i, i) \in A, \tag{1}$$

$$x(\delta^{in}(s)) = 0, \tag{2}$$

$$x(\delta^{out}(t)) = 0, \tag{3}$$

$$x(\delta^{out}(s)) = 1, \tag{4}$$

$$x(\delta^{in}(t)) = 1, \tag{5}$$

$$x(\delta^{out}(i)) - x(\delta^{in}(i)) = 0, \qquad i \in V \setminus \{s, t\}, \tag{6}$$

$$x(A) \le k, \tag{7}$$

$$x(\delta^{out}(i)) \le 1, \qquad i \in V \setminus \{s, t\}, \tag{8}$$

$$x(\delta^{out}(S)) - x(\delta^{out}(j)) \ge 0, \qquad S \subset V, s, t \in S, j \in V \setminus S, \tag{9}$$

$$x_{ij} \in \{0, 1\}, \qquad (i, j) \in A. \tag{10}$$

Here, for any $S \subseteq V$, $\delta^{\text{out}}(S) := \{(i, j) \in A : i \in S, j \in V \setminus S\}$ and $\delta^{\text{in}}(S) := \{(i, j) \in A : i \in V \setminus S, j \in S\}$. For nodes $j \in V$, we write $\delta^{\text{out}}(j)$ and $\delta^{\text{in}}(j)$ instead of $\delta^{\text{out}}(\{j\})$ and $\delta^{\text{in}}(\{j\})$, respectively. Moreover, for any $B \subseteq A$, $x(B) := \sum_{a \in B} x_a$.

Fact 2 *(Dahl and Gouveia [7]). The nonnegativity constraints $x_{ij} \ge 0$ for all $(i, j) \in A$, the Eqs. (2)–(6), and the inequalities*

$$x_{si} - \sum_{j \in V \setminus \{s,t\}} x_{ij} \ge 0 \quad \text{for all } i \in V \setminus \{s, t\}$$

provide a complete linear description of $P_{s,t\text{-}path}^{\le 3}(D)$.

Fact 3 *(Dahl, Foldnes, and Gouveia [6]). The 4-hop constrained walk polytope $\mathcal{W}^{\le 4}(D)$ is determined by the Eqs. (2)–(6), the nonnegativity constraints $x_{ij} \ge 0$ for all $(i, j) \in A$, and the inequalities*

$$\sum_{i \in I} x_{si} + \sum_{j \in J} x_{jt} - \sum_{i \in I, j \in J} x_{ij} \ge 0 \tag{11}$$

for all $I, J \subseteq V \setminus \{s, t\}$.