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Lower bounds for treewidth of product graphs

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ABSTRACT

Two lower bounds for the treewidth of product graphs are presented in terms of the bramble number. The first bound is that the bramble number of the Cartesian product of graphs G_1 and G_2 must be at least the product of the Hadwiger number of G_1 and the PI number of G_2, where the PI number is a new graph parameter introduced in this paper. The second bound is that the bramble number of the strong product of graphs G_1 and G_2 must be at least the product of the Hadwiger number of G_1 and the bramble number of G_2. We also demonstrate applications of the lower bounds.

1. Introduction

The concept of *treewidth* has contributed greatly to pure and algorithmic graph theories in the recent decades. In rough terms, the treewidth of a graph G, denoted by $\mathrm{tw}(G)$, is a graph parameter that measures the proximity of G to a tree. In this paper, we present two lower bounds for the treewidth of product graphs. For this purpose, instead of the treewidth, we use another graph parameter known as the bramble number, which is essentially the same as the treewidth. A *bramble* $\mathcal{B} = \{B_1, \ldots, B_{|\mathcal{B}|}\}$ of G is a collection of the vertex sets of connected subgraphs of G such that any B_i and B_j in $\mathcal{B}$ intersect or are *joined* by an edge. The *order* of $\mathcal{B}$ is the least number of vertices required to cover every B_i in $\mathcal{B}$. In other words, it is the size of a minimum hitting set of $\mathcal{B}$. The *bramble number* of a graph G, denoted by $\mathrm{bn}(G)$, is the maximum order of all brambles of G. Seymour and Thomas [22] showed that $\mathrm{bn}(G) = \mathrm{tw}(G) + 1$ for any graph G. A merit of using a bramble is that a lower bound for the treewidth may be found *constructively*. That is, if we construct a bramble of order greater than k, then the bramble is a certificate of a lower bound k of the treewidth.

1.1. Motivation

Our study was motivated by the following natural question that arises from a study of the inapproximability of the bramble number: is there a graph product operation under which the treewidth of a resulting product graph can be determined only by the treewidths of its factor graphs? We explain how this question relates to the inapproximability by providing a famous example. The *clique number* $\omega(G)$ of a graph G is the size of a maximum clique in G. The *strong product* of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \boxtimes G_2$, is the graph with the vertex set $V_1 \times V_2$, in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if $u_1 = u_2$ or $\{u_1, u_2\} \in E_1$, and $v_1 = v_2$ or $\{v_1, v_2\} \in E_2$ (see Fig. 1 for an example). It is

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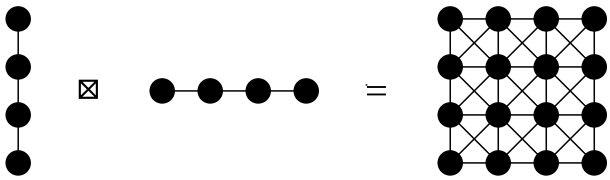

Fig. 1. The strong product of two paths.

not difficult to observe that $\omega(G_1 \boxtimes G_2) = \omega(G_1) \cdot \omega(G_2)$ for any graphs G_1 and G_2. Hence, if we denote the strong power of k copies of G as G', then $\omega(G') = (\omega(G))^k$. Furthermore, it is known that for a given clique C' of G', we can compute a clique C of G with size at least $|C'|^{1/k}$ in polynomial time. This fact can be used to amplify the approximation hardness of the clique number as follows (see [1, Section 6.4]). Assume that there is an α-approximation algorithm $\mathcal{A}$ for the clique number with a constant approximation ratio $\alpha > 1$; that is, for any input, $\mathcal{A}$ outputs a clique of size at least $1/\alpha$ of the maximum. Let $\mathcal{A}'$ be the following (polynomial-time) algorithm:

1. compute the strong power G' of k copies of G;
2. apply $\mathcal{A}$ to G' and obtain a clique C' of size at least $\alpha^{-1} \cdot \omega(G')$;
3. compute a clique C of G with size at least $|C'|^{1/k}$ from C';
4. output C.

From the above discussion, it follows that

$$|C| \geq |C'|^{1/k} \geq (\alpha^{-1} \cdot \omega(G'))^{1/k} = (\alpha^{-1} \cdot \omega(G)^k)^{1/k} = \alpha^{-1/k} \cdot \omega(G).$$

This implies that $\mathcal{A}'$ is an $\alpha^{1/k}$-approximation algorithm for the clique number. Therefore, for any $r > 1$, we can obtain a polynomial-time r-approximation algorithm for the clique number by setting $k \geq \log \alpha / \log r$. That is, $\mathcal{A}'$ is a PTAS for the maximum clique problem. On the other hand, it can be shown using the PCP theorem that such a PTAS does not exist unless P = NP (see [1, Section 6.4]). Hence, there is no constant-factor approximation algorithm for the clique number, unless P = NP. Note that the PCP theorem allows us to have a stronger approximation hardness for clique. See Håstad's result [15] that shows the $n^{1-\epsilon}$ approximation hardness for any $\epsilon > 0$.

The approximability of the treewidth (and thus, that of the bramble number) is well studied. The best known approximation ratio $O\left(\sqrt{\log \text{opt}}\right)$, where opt is the optimum value, was derived by Feige, Hajiaghayi, and Lee [12]. On the other hand, inapproximability is a long-standing open issue. The only known fact is the hardness of additive error approximation. Bodlaender, Gilbert, Hafsteinsson, and Kloks [4] showed that no polynomial-time algorithm $\mathcal{A}$ for the treewidth of a graph G can guarantee $\mathcal{A}(G) \leq \text{tw}(G) + |V(G)|^{\epsilon}$ for any constant $\epsilon < 1$ unless P = NP. It is not known whether the problem admits a PTAS or a constant factor approximation algorithm. If we have a graph product $\otimes$ such that $\text{tw}(G_1 \otimes G_2) = \text{tw}(G_1) \cdot \text{tw}(G_2)$ or at least $\text{tw}(G \otimes G) = (\text{tw}(G))^2$, and we also have a polynomial-time algorithm for computing a tree-decomposition of G with width at most $w^{1/k}$ from a tree-decomposition of $\bigotimes_{1 \leq i \leq k} G$ with width w, then having a constant factor approximation is equivalent to having a PTAS for treewidth. This may help in the study of inapproximability of treewidth.

Quite recently, Austrin, Pitassi, and Wu [2] have proved, assuming the recently introduced Small Set Expansion (SSE) conjecture [19], that approximating the treewidth of a graph in any constant factor is NP-hard. They mentioned that since the status of the SSE conjecture is very uncertain, their inapproximability result should not be taken as the absolute evidence that there is no constant factor approximation for treewidth.

1.2. Our results

Our original motivation was to study the possibility of the existence of a graph product operation $\otimes$ such that $\text{bn}(G_1 \otimes G_2) = \text{bn}(G_1) \cdot \text{bn}(G_2)$ for any graphs G_1 and G_2, or at least, $\text{bn}(G \otimes G) = (\text{bn}(G))^2$ for any graph G. This work presents the first step for this direction in the study of the inapproximability of the bramble number (and treewidth). In this work, we present two lower bounds for the bramble number of product graphs in terms of two related graph parameters. We first show that the bramble number of the Cartesian product of graphs G_1 and G_2 is at least the product of the *PI number* of G_1 and the *Hadwiger number* of G_2; that is, in our terminology, $\text{bn}(G_1 \square G_2) \geq \iota(G_1) \cdot \eta(G_2)$, where $\iota(G)$ and $\eta(G)$ denote the PI number and the Hadwiger number of G, respectively. Next, we show, using a similar argument, that the bramble number of the strong product of graphs G_1 and G_2 is at least the product of the bramble number of G_1 and the Hadwiger number of G_2; that is, $\text{bn}(G_1 \boxtimes G_2) \geq \text{bn}(G_1) \cdot \eta(G_2)$. See Section 2 for the definitions and descriptions of notations used.

We also demonstrate applications of the lower bounds. By applying one lower bound, we determine, with an additive error of 1, the treewidth of the Cartesian product graph of a complete graph and a grid. We also apply the lower bound to the Cartesian product graph of a complete graph and a complete multipartite graph. Unfortunately, our two lower bounds are very weak in the case where both the factor graphs have small treewidth. For example, although $\text{bn}(P_n \square P_n) = n + 1$ and $\text{bn}(P_n \boxtimes P_n) \geq n + 1$, the lower bound functions give $\text{bn}(P_n \square P_n) \geq \iota(P_n) \cdot \eta(P_n) = 2$ and $\text{bn}(P_n \boxtimes P_n) \geq \text{bn}(P_n) \cdot \eta(P_n) = 4$, where P_n is the path of n vertices. On the other hand, these lower bounds work well in the case where one of the two factor