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Perfect domination sets in Cayley graphs

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ABSTRACT

In this paper, we get some results related to perfect domination sets of Cayley graphs. We show that if a Cayley graph $C(\mathcal{A}, X)$ has a perfect dominating set S which is a normal subgroup of $\mathcal{A}$ and whose induced subgraph is F, then there exists an F-bundle projection $p : C(\mathcal{A}, X) \to K_m$ for some positive integer m. As an application, we show that for any positive integer n, the following are equivalent: (a) the hypercube Q_n has a perfect total domination set, (b) $n = 2^m$ for a positive integer m, (c) Q_n is a $2^{n-\log_2 n-1}K_2$-bundle over the complete graph K_n and (d) Q_n is a covering of the complete bipartite graph $K_{n,n}$.

1. Introduction

Let G be a connected finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v. We use $|X|$ for the cardinality of a set X.

A subset S of $V(G)$ is called a *perfect domination set* (PDS) of G if for each $v \in V(G) - S$, there exists a unique element $s \in S$ such that v and s are adjacent. A perfect domination set is efficient if S is independent. A subset S of $V(G)$ is called a *perfect total domination set* (PTDS) of G if for each $v \in V(G)$, there exists a unique element $s \in S$ such that v and s are adjacent. For a graph H, a subset S of $V(G)$ is called an H-PDS if S is a PDS and the subgraph $\langle S\rangle$ induced by S is isomorphic to H. Notice that if H is the null graph, then H-PDS is just an efficient perfect domination set (EPDS), and if $\langle S\rangle$ is isomorphic to the disjoint union of m copies of K_2, then H-PDS is just a PTDS.

A permutation graph was introduced by Chartrand and Harary in [1] as a generalization of the Petersen graph and a graph bundle was obtained by Mohar et al. [6] as a generalization of the covering graph. A permutation graph over a given graph G was introduced by Lee and Sohn [5] as a generalization of both a permutation graph and a graph bundle over a graph. For completeness we recall the definition. Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the edge reverse to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$. Following Gross and Tucker [3] a (*permutation*) *voltage assignment* ϕ of G is a function $\phi : D(G) \to \mathcal{S}_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. Let $C^1(G; \mathcal{S}_n)$ denote the set of all voltage assignments of G. Let F be another graph with $V(F) = \{v_1, v_2, \dots, v_m\}$. For a voltage assignment $\phi \in C^1(G; \mathcal{S}_m)$ of G, we construct a new graph $G\bowtie^{\phi} F$ as follows: $V(G\bowtie^{\phi} F) = V(G) \times V(F)$, and two vertices (u_i, v_h) and (u_j, v_k) are adjacent in $G\bowtie^{\phi} F$ if either $u_iu_j \in D(G)$ and $v_k = \phi(u_iu_j)v_h$ or $u_i = u_j$ and $v_hv_k \in E(F)$. This new graph $G\bowtie^{\phi} F$ is called an *F-permutation graph* over G and the first coordinate projection $p^{\phi} : G\bowtie^{\phi} F \to G$ is called the F-permutation projection. We note that if G is the complete graph K_2 on two vertices, then the F-permutation graph $G\bowtie^{\phi} F$ is just a permutation graph. Note that the set of all graph automorphisms of F forms a subgroup of $\mathcal{S}_m$ and is denoted by Aut (F). If ϕ takes its values in Aut (F), then the *F-permutation graph* $G\bowtie^{\phi} F$

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is just an F-bundle $G \times^{\phi} F$ over G, where the F-permutation projection $p^{\phi} : G \bowtie^{\phi} F \to G$ is the bundle projection. Moreover if F is the null graph of order n, then it is just an n-fold covering G^{ϕ} over G.

Let $\mathcal{A}$ be a group and let $X = \{x_1, \ldots, x_n\}$ be a symmetric generating set for $\mathcal{A}$, i.e., $X^{-1} = \{x^{-1} : x \in X\} = X$. From now on, we will assume that the generating sets considered do not contain the identity element of $\mathcal{A}$. The Cayley graph $C(\mathcal{A}, X)$ is a graph whose vertex set is $\mathcal{A}$, and two vertices g and h are adjacent if and only if $h = gx_i$ for some $x_i \in X$.

In this paper, we get some results related to perfect domination sets of Cayley graphs. In Section 3, we show that if a Cayley graph $C(\mathcal{A}, X)$ has a perfect dominating set S which is a normal subgroup of $\mathcal{A}$ and whose induced subgraph is F, then there exists an F-bundle projection $p : C(\mathcal{A}, X) \to K_m$ for some positive integer m. As an application, in Section 4 we get some equivalent statements for the hypercube Q_n to have a perfect total domination set.

2. Perfect domination sets and permutation graphs

Lemma 1. (a) *Suppose that S_1 and S_2 are two disjoint perfect domination sets of a graph G. Then $|S_1| = |S_2|$ and the set $E(S_1, S_2)$ of all edges between S_1 and S_2 forms a perfect matching in the induced subgraph $\langle S_1 \cup S_2 \rangle$.*

(b) *Suppose that S_1 and S_2 are two perfect domination sets such that the maximum degrees of the induced graphs $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are at most 1 and each isolated vertex of the induced subgraph $\langle S_1 \cap S_2 \rangle$ is either an isolated vertex in both $\langle S_1 \rangle$ and $\langle S_2 \rangle$ or a vertex of degree 1 in both $\langle S_1 \rangle$ and $\langle S_2 \rangle$. Then $|S_1| = |S_2|$. In particular, if both S_1 and S_2 are EDSs or both are PTDSs, then $|S_1| = |S_2|$.*

Proof. (a) is clear. Now, we aim to prove (b). We decompose the set $S_1 \cup S_2$ as follows:

$$(S_1 \setminus S_2) \cup (S_2 \setminus S_1) \cup I \cup ((S_1 \cap S_2) \setminus I),$$

where I is the set of isolated vertices in the induced subgraph $\langle S_1 \cap S_2 \rangle$. For each $s_1 \in S_1 \setminus S_2$, there exists a unique vertex s_2 in S_2 such that s_1 and s_2 are adjacent. If s_2 belongs to $S_1 \cap S_2$, then there is a unique $s_2' \in S_2 \setminus S_1$ adjacent to s_2 by our assumption. Let us define $f : S_1 \setminus S_2 \to S_2 \setminus S_1$ by

$$f(s_1) = \begin{cases} s_2 & \text{if } s_2 \in S_2 \setminus S_1 \\ s_2' & \text{if } s_2 \in S_1 \cap S_2. \end{cases}$$

Then f is injective and hence $|S_1 \setminus S_2| \le |S_2 \setminus S_1|$. This implies that $|S_1| \le |S_2|$. Similarly one can show that $|S_2| \le |S_1|$. This completes the proof. □

Lemma 2. *Let F be a graph and let $S_1, \ldots, S_n$ be n F-PDSs of a graph G which are pairwise mutually disjoint. Then the subgraph H induced by $S_1 \cup \cdots \cup S_n$ is an F-permutation graph over the complete graph K_n.*

Proof. It comes from Lemma 1(a) that the set $E(S_i, S_j)$ forms a perfect matching for any distinct integers $i, j = 1, \ldots, n$. For convenience, we identify the set S_i with the set $N_m = \{1, 2, \ldots, m = |V(F)|\}$. Then the set $E(S_i, S_j)$ induces a permutation $\sigma_{ij} : N_m \to N_m$ for any distinct integers $i, j = 1, \ldots, n$. We define a permutation voltage assignment $\phi : D(K_n) \to \mathcal{S}_m$ by $\phi(v_i v_j) = \sigma_{ij}$. Then the F-permutation graph $K_n \bowtie^{\phi} F$ is isomorphic to H. In fact, for each $i = 1, 2, \ldots, n$, by identifying S_i with $\left(p^{\phi}\right)^{-1}(v_i)$, we can get an isomorphism between H and $K_n \bowtie^{\phi} F$. □

Lemma 3. *Let F, G and H be simple graphs and let $p : G \to H$ be an F-permutation projection. Let S be a PDS of H. Then $p^{-1}(S)$ is a PDS of G. In particular, if S is independent, then $p^{-1}(S)$ is an $|S|F$-PDS of G.*

Proof. Let $v \in V(G) \setminus p^{-1}(S)$. Then $p(v) \in V(H) \setminus S$. Since S is a PDS of H, there exists a unique $s \in S$ such that s is adjacent to $p(v)$. It comes form the definition of F-permutation projection that the edge set $E(p^{-1}(s), p^{-1}(p(v)))$ induces a bijection $\sigma \in \mathcal{S}_{|V(F)|}$. In fact, if the voltage assignment associated with the F-permutation projection p is $\phi : D(H) \to \mathcal{S}_{|V(F)|}$, then $\phi(s\,p(v)) = \sigma$. Since $v \in p^{-1}(p(v))$, there exists a unique $\tilde{s} \in p^{-1}(s)$ such that v and $\tilde{s}$ are adjacent in G. This says that $p^{-1}(S)$ is a domination set of G. In order to show perfectness, we assume that v is adjacent to $\tilde{s}' \in p^{-1}(S)$. Then $p(v)$ and $p(\tilde{s}')$ are adjacent in H. Since S is a PDS of H and $p(\tilde{s}') \in S$, $p(\tilde{s}')$ must be s. This implies that $\tilde{s}'$ must be $\tilde{s}$ and hence $p^{-1}(S)$ is a perfect domination set of G.

Assume that S is independent. Since $p^{-1}(s)$ is isomorphic to F for each $s \in S$ and S is independent, the subgraph induced by $p^{-1}(S)$ is isomorphic to the union of $|S|$ copies of F. Now it comes from the first statement that $p^{-1}(S)$ is an $|S|F$-PDS of G. □

Corollary 4. *For two simple graphs F and G and for a positive integer n, G is an F-permutation graph over the complete graph K_n if and only if G has a vertex partition $\{S_1, \ldots, S_n\}$ such that S_i is an F-PDS for each $i = 1, 2, \ldots, n$.*

Proof. The sufficiency follows from Lemma 2. For the necessity, let $p : G \to K_n$ be an F-permutation projection. Since for each $v \in V(K_n)$, $\{v\}$ is an independent perfect domination set of K_n, $p^{-1}(v)$ is an F-PDS of G by Lemma 3. Since p is an F-permutation projection, $p^{-1}(u)$ and $p^{-1}(v)$ are disjoint for any two distinct vertices u and v of K_n and hence $\{p^{-1}(v) : v \in V(K_n)\}$ is a partition of $V(G)$. This completes the proof. □