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Note

On the intersection of all critical sets of a unicyclic graph

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ABSTRACT

A set $S \subseteq V$ is *independent* in a graph $G = (V, E)$ if no two vertices from S are adjacent. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set, while $\mu(G)$ is the cardinality of a maximum matching in G. If $\alpha(G) + \mu(G) = |V|$, then G is a *König–Egerváry graph*. The number

$$d(G) = \max\{|A| - |N(A)| : A \subseteq V\}$$

is the *critical difference* of G (Zhang, 1990) [22], where $N(A) = \{v : v \in V, N(v) \cap A \neq \emptyset\}$.

By core(G) (corona(G)) we denote the intersection (union, respectively) of all maximum independent sets, and by ker(G) we mean the intersection of all critical sets. A connected graph having only one cycle is called unicyclic.

It is known that the relation ker$(G) \subseteq$ core(G) holds for every graph G (Levit, 2012) [14], while the equality is true for bipartite graphs (Levit, 2013) [15]. For König–Egerváry unicyclic graphs, the difference $|\text{core}(G)| - |\ker(G)|$ may equal any non-negative integer.

In this paper we prove that if G is a non-König–Egerváry unicyclic graph, then: (i) ker(G) = core(G) and (ii) $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1$. Pay attention that $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$ holds for every König–Egerváry graph (Levit, 2011) [11].

1. Introduction

Throughout this paper G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting the edge set W, for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$. If $A, B \subseteq V(G)$, then (A, B) stands for the set $\{ab : a \in A, b \in B, ab \in E(G)\}$.

The *neighborhood* $N(v)$ of a vertex $v \in V(G)$ is the set $\{w : w \in V(G) \text{ and } vw \in E(G)\}$, while the *closed neighborhood* $N[v]$ of $v \in V(G)$ is the set $N(v) \cup \{v\}$; in order to avoid ambiguity, we also use $N_G(v)$ instead of $N(v)$. The *neighborhood* $N(A)$ of $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$. We may also use $N_G(A)$ and $N_G[A]$ for clarity when referring to neighborhoods in a graph G.

By C_n, K_n we mean the chordless cycle on $n \geq 4$ vertices, and respectively the complete graph on $n \geq 1$ vertices.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by Ind(G) we mean the family of all the independent sets of G. An independent set of maximum size is a *maximum independent set* of G, and the *independence number* $\alpha(G)$ is the cardinality of a maximum independent set of G.

A *matching* is a set M of pairwise non-incident edges of G. For $xy \in M$, we say that the vertices x and y are *matched* by M. The matching M is *from A into B* if every vertex of A is matched by M to some vertex from B. A matching of maximum

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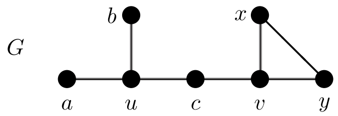

Fig. 1. A König–Egerváry graph with $\alpha(G) = |\{a, b, c, x\}|$ and $\mu(G) = |\{au, cv, xy\}|$.

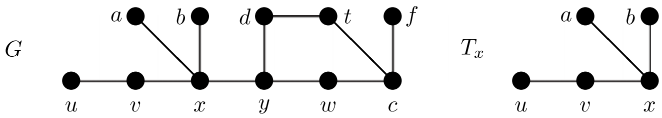

Fig. 2. G is a unicyclic graph with $V(C) = \{y, d, t, c, w\}$ and $N(C) = \{x, f\}$.

cardinality is a *maximum matching*, and its size is denoted by $\mu(G)$. The matching number is also denoted by $\alpha'(G)$, since it is the independence number of the line graph. Other standard notations from graph theory may be found in [20].

Let $\mathrm{core}(G) = \bigcap\{S : S \in \Omega(G)\}$ [7], and $\mathrm{corona}(G) = \bigcup\{S : S \in \Omega(G)\}$ [1], where $\Omega(G) = \{S : S$ *is a maximum independent set of* $G\}$. The problem of whether $\mathrm{core}(G) \neq \emptyset$ is **NP**-hard [1].

Theorem 1.1 (*[1]*). *For every* $S \in \Omega(G)$, *there is a matching from* $S - \mathrm{core}(G)$ *into* $\mathrm{corona}(G) - S$.

Notice that $\alpha(G) \leq \alpha(G-e) \leq \alpha(G)+1$ holds for each edge e. An edge $e \in E(G)$ is α-*critical* whenever $\alpha(G-e) > \alpha(G)$. An edge $e \in E(G)$ is μ-*critical* provided $\mu(G-e) < \mu(G)$.

For $X \subseteq V(G)$, the number $|X| - |N(X)|$ is the *difference* of X, denoted by $d(X)$. The *critical difference* $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. The number $\max\{d(I) : I \in \mathrm{Ind}(G)\}$ is the *critical independence difference* of G, denoted by $\mathrm{id}(G)$. Clearly, $d(G) \geq \mathrm{id}(G)$ is true for every graph G. It was shown in [22] that $d(G) = \mathrm{id}(G)$ holds for every graph G. If A is an independent set with difference $\mathrm{id}(G)$, then A is a *critical independent set* [22].

For a graph G, let denote $\ker(G) = \bigcap\{S : S$ *is a critical set*$\}$ [14].

Theorem 1.2. (i) [14] $\ker(G)$ *is the unique minimal critical (independent) set of* G *and* $\ker(G) \subseteq \mathrm{core}(G)$;
(ii) [15] $\ker(G) = \mathrm{core}(G)$, *whenever* G *is bipartite.*

Some other structural properties of $\ker(G)$ may be found in [17].

It is well-known that

$$\lfloor n/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n$$

hold for any graph G with n vertices. If $\alpha(G) + \mu(G) = n$, then G is a *König–Egerváry graph* [4,19]. Several properties of König–Egerváry graphs are presented in [12,16].

According to a celebrated result of König and Egerváry, every bipartite graph is a König–Egerváry graph [5,6]. This class also includes non-bipartite graphs (see, for example, the graph G in Fig. 1).

Theorem 1.3 (*[8]*). *If* G *is a König–Egerváry graph, then every maximum matching matches* $N(\mathrm{core}(G))$ *into* $\mathrm{core}(G)$.

The graph G is *unicyclic* if it is connected and has a unique cycle, which we denote by $C = (V(C), E(C))$. Let $N_1(C) = \{v : v \in V(G) - V(C), N(v) \cap V(C) \neq \emptyset\}$, and $T_x = (V_x, E_x)$ be the maximum subtree of $G - xy$ containing x, where $x \in N_1(C)$ and $y \in V(C)$ (see, for an example, Fig. 2).

The following result shows that a unicyclic graph is either a König–Egerváry graph or each edge of its cycle is α-critical.

Lemma 1.4 (*[13]*). *If* G *is a unicyclic graph of order* n, *then*
(i) $n - 1 \leq \alpha(G) + \mu(G) \leq n$;
(ii) $n - 1 = \alpha(G) + \mu(G)$ *if and only if each edge of the unique cycle is* α-*critical.*

For more properties of critical edges in König–Egerváry graphs, see [9].

Theorem 1.5 (*[13]*). *Let* G *be a unicyclic non-König–Egerváry graph. Then the following assertions are true:*
(i) *each* $W \in \Omega(T_x)$ *can be enlarged to some* $S \in \Omega(G)$;
(ii) $S \cap V(T_x) \in \Omega(T_x)$ *for every* $S \in \Omega(G)$;
(iii) $\mathrm{core}(G) = \bigcup\{\mathrm{core}(T_x) : x \in N_1(C)\}$.

Unicyclic graphs keep enjoying plenty of interest, as one can see, for instance, in [2,3,10,18,21]. Our decision of concentrating on unicyclic graphs is based on their striking similarities to König–Egerváry graphs. For instance, unicyclic non-König–Egerváry graphs satisfy

- the fact that $|\mathrm{core}(G)| \neq 1$, like bipartite graphs [7];
- the equality $\ker(G) = \mathrm{core}(G)$, like bipartite graphs [15];
- the property that $\mathrm{core}(G)$ is critical, like König–Egerváry graphs [12].

In this paper we analyze various relationships between several parameters of a unicyclic graph G, namely, $\mathrm{core}(G)$, $\mathrm{corona}(G)$, and $\ker(G)$.