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Note

The matching energy of graphs with given parameters

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ABSTRACT

Gutman and Wagner [I. Gutman, S. Wagner, The matching energy of a graph, Discrete Appl. Math. 160 (2012) 2177–2187] defined the matching energy of a graph and gave some properties and asymptotic results of the matching energy. In this paper, we characterize the connected graph G with the connectivity κ (resp. chromatic number χ) which has the maximum matching energy.

1. Introduction

In this paper we are concerned with undirected simple graphs (without multiple edges or loops). Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. Denote by $\kappa(G)$, $\chi(G)$, and $\varepsilon(G)$ the connectivity (i.e., vertex connectivity), the chromatic number and the number of edges of a graph G, respectively. By $G - u$ we denote the induced subgraph obtained from G by deleting vertex u together with its incident edges and by $G - e$ the edge-induced subgraph obtained from G by deleting edge e. Let K_n be the complete graph and $K_{n_1,n_2,\dots,n_k}$ the complete k-partite graph with $n = \sum_{i=1}^{k} n_i$ vertices, whose vertex set is partitioned into k parts: $V_1, V_2, \dots, V_k$, of cardinalities $n_1, n_2, \dots, n_k$, and the edge joins two vertices if and only if they belong to different parts. Write $T_{m,n}$ for the Turán graph, that is a complete m-partite graph in which each part has either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ vertices. The vertex-disjoint union of two graphs G and H is denoted by $G \cup H$. Let $G \vee H$ be the graph obtained from $G \cup H$ by adding all possible edges from vertices of G to vertices of H, i.e., $G \vee H = \overline{\overline{G} \cup \overline{H}}$, where $\overline{G}$ is the complement of G. By $m(G, k)$ we denote the number of k-matchings (= the number of selections of k-independent edges = the number of k-element independent edge sets) of a graph G. Specifically, $m(G, 1) = \varepsilon(G)$ and $m(G, k) = 0$ for $k > n/2$. It is both consistent and convenient to define $m(G, 0) = 1$. The Hosoya index $Z(G)$ is defined as the total number of the matchings. That is,

$$Z(G) = m(G, 0) + m(G, 1) + \cdots + m\left(G, \left[\frac{n}{2}\right]\right).$$

The matching polynomial of a graph G with n vertices is defined as

$$\alpha(G) = \alpha(G, \lambda) = \sum_{k \ge 0} (-1)^k m(G, k) \lambda^{n-2k}, \tag{1}$$

where $m(G, 0) = 1$ and $m(G, k) \ge 0$ for all $k = 1, 2, \dots, [n/2]$. Its theory is well elaborated [2,4–6]. This expression for $\alpha(G, \lambda)$ induces a quasi-order relation (i.e., reflexive and transitive relation) on the set of all graphs with n vertices: If G and

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H are two graphs with matching polynomial in the form (1), then the quasi-order $\succeq$ is defined by

$$G \succeq H \Longleftrightarrow m(G,k) \geq m(H,k) \quad \text{for all } k = 0, 1, \ldots, [n/2].$$

If $G \succeq H$ and there exists some k such that $m(G,k) > m(H,k)$, then we write $G \succ H$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G, i.e., the eigenvalues of its (0, 1)-adjacency matrix [3]. The energy of the graph G is then defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The theory of graph energy is well developed nowadays; its details can be found in the book [10] and reviews [7,8].

Gutman and Wagner [9] defined the matching energy of a graph G with n vertices, denoted by $ME(G)$, as

$$ME = ME(G) = \frac{2}{\pi} \int_0^{\infty} x^{-2} \ln \left[\sum_{k \geq 0} m(G,k) x^{2k} \right] dx. \tag{2}$$

They also pointed out that: if the graph G is a forest, its matching energy coincides with its energy.

The integral on the right hand side of Eq. (2) is increasing in all the coefficients $m(G,k)$. From this fact, one can readily deduce the extremal graphs for the matching energy. Since

$$m(G,k) = m(G-e,k) + m(G-u-v,k-1) \tag{3}$$

holds for any edge e of the graph G, connecting the vertices u and v [4–6], we see that $m(G,k)$ can only increase when edges are added to a graph (and at least for $m(G,1)$, the number increases strictly when this is done). From the definition of the matching energy, it is clear that: $G \succeq H \Longrightarrow ME(G) \geq ME(H)$ and $G \succ H \Longrightarrow ME(G) > ME(H)$. This increased property of ME has been successfully applied in the study of the extremal values of the matching energy over a significant class of graphs, see the following results.

Theorem 1.1 ([9]). *Let G be a graph and e an edge of G. Let $G-e$ be the subgraph obtained by deleting edge e from G, but keeping all the vertices of G. Then*

$$ME(G-e) < ME(G).$$

Lemma 1.2 ([9]). *Among all graphs with n vertices, the empty graph E_n without edges and the complete graph K_n have, respectively, minimum and maximum matching energy.*

Lemma 1.3 ([9]). *The connected graph with n vertices having minimum matching energy is the star S_n.*

Lemma 1.4 ([9]). *The bipartite graph with n vertices having maximum matching energy is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

For the k-matching $m(G,k)$ of a graph G, if $k = 1$, then $m(G,k)$ is the number of edges of G; if $k = \frac{n}{2}$ (n is even), then $m(G,k)$ is the number of perfect matchings $\phi(G)$ of G. For this, we review briefly some known results.

Theorem 1.5 ([1]). *If G is a complete m-partite graph with n vertices, then $\varepsilon(G) \leq \varepsilon(T_{m,n})$, with equality only if $G \cong T_{m,n}$.*

Theorem 1.6 ([11]). *Let G be a connected graph of order $2n$ with connectivity $\kappa(G) \leq k$. Then the number of perfect matchings satisfies $\phi(G) \leq k[(2n-3)!!]$.*

Theorem 1.7 ([11]). *Let G be a graph of order $2n$ with the chromatic number $\chi(G) \leq k$. Then the number of perfect matchings satisfies $\phi(G) \leq \phi(T_{k,2n})$.*

Let $V_{n,k}$ be the set of connected graphs of order n with connectivity at most k. Xu, Li and Zhong [13] proved that $(K_1 \cup K_{n-\kappa-1}) \vee K_\kappa$ is a unique graph in $V_{n,k}$ with maximal Hosoya index. Xu [12] also proved that: for all connected graphs of order n with chromatic number χ, Turán graph $T_{\chi,n}$ has the maximal Hosoya index.

In this paper, in addition to the results in [9,11–13], we proved that: if G is a connected graph of order n with connectivity κ, then for each k, $m(G,k) \leq m[(K_1 \cup K_{n-\kappa-1}) \vee K_\kappa, k]$, and if G is a connected graph of order n with chromatic number χ, then for each k, $m(G,k) \leq m(T_{\chi,n}, k)$. These results characterize the connected graph G with the connectivity κ (resp. chromatic number $\chi(G)$) which has the maximum matching energy.

2. Main results

The following result generalizes the result of Theorem 1.5 and the one in Xu [12].

Lemma 2.1. *Let G be a complete m-partite graph with n vertices, $0 \leq k \leq \lfloor n/2 \rfloor$. Then*

$$m(G,k) \leq m(T_{m,n}, k).$$

The equality holds if and only if $G \cong T_{m,n}$.