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An algorithm for finding the vertices of the k-additive monotone core

Pedro Miranda [a,*], Michel Grabisch [b]

[a] Universidad Complutense de Madrid, Plaza de Ciencias, 3, 28040 Madrid, Spain

[b] Université Paris I-Panthéon-Sorbonne, 106-112 Bd. de l'Hôpital, 75013 Paris, France

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ABSTRACT

Given a capacity, the set of dominating k-additive capacities is a convex polytope called the k-additive monotone core; thus, it is defined by its vertices. In this paper, we deal with the problem of deriving a procedure to obtain such vertices in the line of the results of Shapley and Ichiishi for the additive case. We propose an algorithm to determine the vertices of the n-additive monotone core and we explore the possible translations for the k-additive case.

1. Introduction

One of the main problems of cooperative game theory is to define a solution of a game v, that is, supposing that all players join the grand coalition N, an *efficient pay-off vector* or *pre-imputation* to each player represents a sharing of the total worth of the game $v(N)$. In the case of finite games of n players, a pre-imputation can be written as an n-tuple $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i = v(N)$. Of course, some rationality criterion should prevail when defining the sharing.

In this respect, the core is perhaps the most popular solution of a game. It is defined as the set of pre-imputations x on N such that

$$\sum_{i\in A} x_i \geq v(A), \quad \forall A \subseteq N,\ A \neq \emptyset \quad \text{and} \quad \sum_{i=1}^{n} x_i = v(N).$$

It is a well-known fact that the core is nonempty if and only if the game is balanced [1]. For games with an empty core, it is necessary to give an alternative solution. In this sense, many possibilities have been proposed in the literature, such as the dominance core stable sets, the Shapley index, the Banzhaf index, the ϵ-core, the kernel, the nucleolus, etc. (see e.g. [8]).

On the other hand, Grabisch has defined in [10] the concept of k-additive capacities (capacities are monotone games). These capacities generalize the concept of probability and they fill the gap between probabilities and general capacities. Moreover, as they are defined in terms of the Möbius transform and this transform can be applied to the characteristic function of any game (not necessarily monotone), the concept of k-additivity can be extended to games as well.

In a previous paper [19], we have defined the so-called k-additive core. The basic idea is to remark that an imputation is nothing else than an additive game and, if the core is empty, we may allow to search for games more general than additive ones, namely k-additive games, dominating the game. We have presented a generalization of the concept of balanced

* Corresponding author. Tel.: +34 91 394 44 19; fax: +34 91 394 46 06.
E-mail addresses: pmiranda@mat.ucm.es (P. Miranda), michel.grabisch@univ-paris1.fr (M. Grabisch).

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games, the k-balanced games; these games are those admitting a dominating k-additive game and no dominating $(k-1)$-additive game. In that paper it is also defined a generalization of the concept of pre-imputation, the k-imputation; from a k-imputation, we have proposed a procedure to define a classical pre-imputation based on the pessimistic criterion.

In [19] we have seen that for general games, any game is either balanced or 2-balanced. Moreover, while the core is a polytope whose vertices have been obtained by Shapley [25] and Ichiishi [15] for convex games, the 2-additive core is not a polytope but an unbounded convex polyhedron [13].

On the other hand, when dealing with capacities, it makes sense to study the k-additive monotone core, that consists in the set of capacities dominating the capacity; it can be easily seen that the k-additive monotone core is a convex polytope, whence it can be described through its vertices. The aim of this paper is to study these vertices.

Moreover, there are other fields in which it is interesting to find the set of probabilities dominating a capacity. For instance, Dempster [6] and Shafer [24] have proposed a representation of uncertainty based on a "lower probability" or "degree of belief", respectively, to every event. Their model needs a lower probability function, usually non-additive but having a weaker property: it is a belief function [24]. This requirement is perfectly justified in some situations (see [6]). The general form of lower probabilities has been studied by several authors (see e.g. [30,31]). Moreover, in many decision problems, in which we have not enough information, decision makers often feel that they are only able to assign an interval value for the probability of events. In other words, they do not know the real probability distribution but there exists a set of probabilities compatible with the available information. Let us call this set of all compatible probabilities $\mathcal{P}_1$ and let us define $\mu := \inf_{P\in\mathcal{P}_1} P$; then, μ is a capacity (but not necessarily a belief function [29]); μ is called "coherent lower probability", and it is the natural "lower probability function". Of course, if P' is a probability measure dominating μ, it is clear that $E_{P'}(f) \geq \int f d\mu$, for any function f, where $\int f d\mu$ represents Choquet integral [3]. Chateauneuf and Jaffray use this fact and that $\mu \leq P,\ \forall P \in \mathcal{P}_1$ in [2] to obtain an easy method for computing a lower bound of $\inf_{P\in\mathcal{P}_1} E_P(f)$ when μ satisfies some additional conditions (namely μ is 2-monotone). Their method is based on obtaining the set of all probability distributions dominating μ. The same can be done for obtaining an upper bound. In this case, we can find a similar motivation for studying the set of all k-additive capacities dominating a capacity.

The paper is organized as follows. In the next section, we give the basic concepts about k-additive capacities and about the set of dominating probabilities. Section 3 is devoted to characterize the vertices for the n-additive case and, in Section 4, we deal with possible generalizations for the k-additive case. In Section 5, we outlined the case of dominated k-additive capacities. We finish with the conclusions and open problems.

2. Basic concepts

We will use the following notation throughout the paper: we suppose a finite universal set with n elements, $N = \{1, \ldots, n\}$. Subsets of N are denoted by capital letters A, B, and so on. The set of subsets of N is denoted by $\mathcal{P}(N)$, while the set of subsets whose cardinality is less than or equal to k is denoted by $\mathcal{P}^k(N)$.

Definition 1 (*[21]*)**.** A **game** over N is a mapping $v : \mathcal{P}(N) \to \mathbb{R}$ (called **characteristic function**) satisfying $v(\emptyset) = 0$.

If, in addition,

1. v satisfies $v(A) \leq v(B)$ whenever $A \subseteq B$, the game v is said to be **monotone**;
2. v satisfies $v(A \cup B) = v(A) + v(B)$ whenever $A, B \subseteq N, A \cap B = \emptyset$, the game is said to be **additive**;
3. v satisfies $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$, for all $A, B \subseteq N$, the game is said to be **convex**.

From the point of view of game theory, for any $A \subseteq N$, the value $v(A)$ represents the minimum asset the coalition of players A will win if the game is played, whatever the remaining players may do, i.e., $v(A)$ is the payoff that coalition A can guarantee for itself. We will denote by $\mathcal{G}(N)$ the set of all games on N.

Definition 2. A **non-additive measure** [7] or **capacity** [3] or **fuzzy measure** [27] μ over N is a monotone game with $\mu(N) = 1$.

Consider a monotone game different from the trivial game defined by $v(A) = 0,\ \forall A \subseteq N$. In this case, we can divide all the values of v by $v(N)$ so that we obtain a new game μ equivalent to v. Then, μ is a capacity and we conclude that any monotone game can be equivalently represented by a capacity. Observe that the set of all capacities on N is a convex polytope, that we will denote $\mathcal{FM}(N)$.

There are other set functions that can be used to equivalently represent a game. We will need in this paper the so-called Möbius transform.

Definition 3 (*[23,14]*)**.** Let v be a game on N. The **Möbius transform** (or **dividends**) of v is a set function on N defined by

$$m_v(A) := \sum_{B\subseteq A} (-1)^{|A\setminus B|} v(B), \quad \forall A \subseteq N.$$