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A conditional edge connectivity of double-orbit networks☆

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HIGHLIGHTS

- The double-orbit graphs with two orbits of the same size is a generalization of vertex transitive graphs, which contains many classic network models.
- The paper characterizes the non-super restricted edge connected double-orbit graphs with two orbits of the same size and girth at least 5.
- The result implies several corollaries on Bi-Cayley graphs, mix-Cayley graphs, half vertex transitive graphs and edge transitive regular graphs.

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ABSTRACT

The edge connectivity is a kind of classic measure of fault tolerance of networks. It is well known that the edge-connectivity of a simple, connected, vertex transitive graph attains its regular degree. It is then natural to consider the relationship between the edge connectivity and the number of orbits of its automorphism group. The double-orbit graphs with two orbits of the same size is a generalization of vertex transitive networks, which contains several classic network models. In this note, we obtain a sufficient condition for such double-orbit graphs to be super-λ'.

1. Introduction

Multiprocessor systems have many advantages over their uniprocessor counterparts, such as high performance and reliability, better reconfigurability and scalability. With the development of VLSI technology and software technology, multiprocessor systems with hundreds of thousands of processors have become available. With the continuous increase in the size of multiprocessor systems, the complexity of a system can adversely affect its fault tolerance and reliability. To the design and maintenance purpose of multiprocessor systems, appropriate measures of reliability should be found. A multiprocessor system can be modeled as a simple connected graph ($G = (V, E)$) with processors and links between processors as vertices and edges, respectively. The graph is called the interconnection network of this multiprocessor system. Thus, graph parameters of its interconnection network (network for short) can be used to measure the reliability of a multiprocessor system.

A well-known model [1] is a network in which nodes are reliable while links may fail independently with the same probability $\rho \in (0, 1)$. One measure of the network reliability is the probability $P(G, \rho)$ of G being disconnected:

$$P(G, \rho) = \sum_{i=\lambda(G)}^{|E|} m_i(G)\rho^i(1-\rho)^{m-i},$$

where $m_i(G)$ is the number of edge cuts of size i and $\lambda(G)$ is the edge connectivity of G. Clearly, the smaller $P(G, \rho)$ is, the more reliable the network is. But in general, to determine $P(G, \rho)$ is difficult [1,2].

Throughout the paper, we assume the graphs considered are simple. Denote by $\Omega(n, e)$ the set of graphs with n vertices and e edges. To minimize $P(G, \rho)$ in $\Omega(n, e)$ when is sufficiently small, the edge connectivity plays an important role. In fact, Bauer et al. [3] showed that for $G_1, G_2 \in \Omega(n, e)$, if $\lambda(G_1) > \lambda(G_2)$, then $P(G_1, \rho) < P(G_2, \rho)$ when is sufficiently small. So in network design, we expect $\lambda(G)$ to be as large as possible. It is well known that $\lambda(G) \le \delta(G)$ holds for any graph G, where $\delta(G)$ is the minimum degree of G. So, the graph G with $\lambda(G) = \delta(G)$ is naturally named as an optimally edge connected graph (or simply, an optimally-graph).

For further study, one of them is the restricted edge-connectivity which was proposed by Esfahanian and Hakimi [4].

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Since in some large interconnection networks the degree of each vertex is large, there is a very small possibility that all the neighbors of a vertex fail simultaneously as we mentioned above. So it is safe to assume that all the neighbors of any vertex will not fail at the same time (this is the restriction). Thus, the restricted edge-connectivity is a more accurate measure of reliability of these interconnection networks. The restricted edge-connectivity of many interconnection networks have been shown to be about twice their traditional edge-connectivity. Thus, restricted edge-connectivity analysis can improve the fault tolerance ability of some interconnection networks theoretically.

It is well known that the transitivity plays an important role in network design. The Cayley graph as a kind vertex transitive graphs has been studied by many authors. In fact, many classic network models (such as hypercubes and star-graphs) are bipartite, and then they are can be seen double-orbit graphs with two orbits of the same size. The double-orbit graphs can be seen a generalization of the vertex transitive graphs. In this paper, we discuss the restricted edge-connectivity of double-orbit graphs that generalizes such results on the classic network models.

The rest of this paper is organized as follows: In Section 2, we state the definitions of double-orbit graphs and several related family of graphs, and provide some preliminaries. Section 3 discusses the super restricted edge connectivity of double-orbit graphs.

2. Preliminaries and terminologies

Unless stated otherwise, we follow Bondy and Murty [5] for terminology and definitions. We consider finite, undirected and simple connected graphs with vertex set $V(G)$ and edge set $E(G)$. We use $d_G(v)$ to denote the degree of vertex $v \in V(G)$ of G. For $X \subset V(G)$, we use $G[X]$ to denote the subgraph induced by X. Let G_1 and G_2 be two graphs. The union $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. An edge set F of G is called an *edge-cut* if $G-F$ is disconnected. The *edge-connectivity* $\lambda(G)$ of a graph G is the minimum cardinality over all edge-cuts of G. A graph G is called *maximally edge connected* or simply *max-*λ if $\lambda(G) = \delta(G)$, see [6] for the details. Furthermore, we call a graph G is *super edge connected* or simply *super-*λ if G is max-λ and every minimum edge-cut set of G isolates one vertex. For the studies of super-λ graphs, we suggest readers to refer to [6,7].

The concept of restricted edge-connectivity is also one kind of conditional edge-connectivity proposed by Harary in [8]. An edge set F is a *restricted edge-cut* of G if $G-F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts is the *restricted edge-connectivity* of G, denoted by $\lambda'(G)$. Esfahanian in [4] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ exists and $\lambda(G) \leq \lambda'(G) \leq \xi(G)$, where $\xi(G) = \min\{d_G(u) + d_G(v) - 2 : uv \in E(G)\}$ is the minimum edge degree of G (for $uv \in E(G)$, we call $d_G(u) + d_G(v) - 2$ the *edge degree* of uv). A graph G with $\lambda'(G) = \xi(G)$ is called λ'*-optimal*. The studies of λ'-optimal graphs can be found in [7,9–11]. An λ'-optimal graph is called *super restricted edge-connected*, or simply *super-*λ', if every minimum edge cut isolates an edge.

A graph G is said to be *vertex transitive* if for every two vertices u and v of G, there exists an automorphism $g \in Aut(G)$, such that $g(u) = v$. For the studies of the connectedness of vertex transitive graphs, we suggest readers to refer to [12–14]. Let $x \in G$, we call the set $\{x^g : g \in Aut(G)\}$ an orbit of $Aut(G)$. If no confusion, we directly call an orbit of $Aut(G)$ an orbit of G. Let W be a subgroup of the symmetric group over a set S. We say that W acts transitively on a subset T of S if for any $h, l \in T$, there exists a permutation $\varphi \in W$ with $\varphi(h) = l$. Clearly, the automorphism group $Aut(G)$ acts transitively on each orbit of G. A graph G is called *double-orbit graph* if G contains exactly two orbits.

We next introduce several generalizations of Cayley graphs which are double-orbit graphs with two orbits of the same size.

Definition 1. For a group $\mathcal{G}$, let S be a subset of $\mathcal{G}$ such that $1_{\mathcal{G}} \notin S$ and $S^{-1} = S$. The *Cayley graph* $C(\mathcal{G}, S)$ is the graph with vertex set $\mathcal{G}$ and edge set $\{(g, sg) : g \in \mathcal{G}, s \in S\}$.

Xu et al. in [15] defined half vertex transitive graphs and Bi-Cayley graphs as follows:

Definition 2. A bipartite graph G with bipartition $X_1 \cup X_2$ is called *half vertex transitive* if $Aut(G)$ acts transitively both on X_1 and X_2.

Definition 3. For a group $\mathcal{G}$, let S be a subset of $\mathcal{G}$. The *Bi-Cayley graph* $BC(\mathcal{G}, S)$ is the graph with vertex set $\mathcal{G} \times \{0, 1\}$ and edge set $\{((g, 0), (sg, 1)) : g \in \mathcal{G}, s \in S\}$.

From the definitions we can see that half vertex transitive graphs include Bi-Cayley graphs. Chen et al. [16] defined mixed Cayley graphs as follows:

Definition 4. Let $\mathcal{G}$ be a finite group, S_0, S_1, S_2 be subsets of $\mathcal{G}$ such that $1_{\mathcal{G}} \notin S_i$ for $i = 0, 1$. The *mixed Cayley graph* $X = MC(\mathcal{G}, S_0, S_1, S_2)$ has vertex set $V(X) = \mathcal{G} \times \{0, 1\}$ and edge set $E(X) = E_0 \cup E_1 \cup E_2$, where $E_i = \{\{((g, i), (s_i g, i))\} : g \in \mathcal{G}, s_i \in S_i\}$ for $i = 0, 1$ and $E_2 = \{\{((g, 0), (s_2 g, 1))\} : g \in \mathcal{G}, s_2 \in S_2\}$.

Clearly, half vertex transitive graphs and mixed Cayley graphs have at most two orbits.

Mader [12] proved that the edge-connectivity of a vertex transitive graph attains its regular degree. It is then natural to consider the relation between the edge connectivity and the number of orbits. In [17], Liu and Meng characterized the 3-regular and 4-regular max-λ double-orbit graphs and also reported a sufficient condition for k-regular double-orbit graphs to be max-λ, for $k > 4$. Later, the authors [18] characterized the 3-regular λ'-optimal double-orbit graphs and given a sufficient condition for the k-regular double-orbit graphs to be λ'-optimal for $k \geq 4$. In [19], W. Yang et al. studied the edge connectivity and restricted edge-connectivity of double-orbit graphs with two orbits of the same size. In this note, we obtain a sufficient condition for such double-orbit graphs to be super-λ'. Furthermore, by applying our results we obtain some results on half vertex transitive graphs, Bi-Cayley graphs and mixed Cayley graphs.

3. Main results

A restricted edge cut F of G is called a λ'-cut if $|F| = \lambda'(G)$. It is easy to see that for any λ'-cut F, $G - F$ has exactly two connected non-trivial components. Let A be a proper subset of V. We denote $\omega(A)$ the set of edges with exactly one end vertex in A. If $\omega(A)$ is a λ'-cut of G, then A is called a λ'*-fragment* of G. It is clear that if A is a λ'-fragment of G, then so is $\bar{A}$. A λ'-fragment B is called a λ'*-atom* of G if $|B| = r(G)$. A λ'-fragment C is called a *strict* λ'*-fragment* if $3 \leq |C| \leq |V(G)|$. If G contains strict λ'-fragments, then the ones with smallest cardinality are called λ'*-superatoms.*

In what follows, we assume that $G = (G_1, G_2, (V_1, V_2))$ be a connected double-orbit graph with two orbits V_1 and V_2 such that $|V_1| = |V_2| \geq 3$, where $G_i = G[V_i]$ is a k_i-regular vertex transitive subgraph of G for $i = 1, 2$ and $(V_1, V_2) = G - E(G_1) \cup E(G_2)$ is a d-regular bipartite double-orbit subgraph of G (from now on, we use the symbol d to denote the regularity). We denote $g(G)$ the girth of G. The following result can be found in [19].

Lemma 1. *Let* $G = (G_1, G_2, (V_1, V_2))$ *be a connected double-orbit graph with two orbits* V_1 *and* V_2 *such that* $|V_1| = |V_2| \geq 3$*. If* $g(G) \geq 5$*, then* G *is* λ'*-optimal.*

Similarly as the definition of $\xi(G)$, we define $\xi_m(G) = \min\{|\omega(U)| : |U| = m, m \geq 3 \text{ and } G[U] \text{ is connected}\}$, where $\omega(U)$ is the set of edges with exactly one end vertex in U and $G[U]$ is