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Streamline topologies of two-dimensional peristaltic flow and their bifurcations

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ABSTRACT

Streamline patterns and their local and global bifurcations in a two-dimensional planar and axisymmetric peristaltic flow for an incompressible Newtonian fluid have been investigated. An analytical solution for the stream-function is found under a long-wavelength and low-Reynolds number approximation. The problem is solved in a moving coordinate system where a system of nonlinear autonomous differential equations can be established for the particle paths. Local bifurcations and their topological changes are inspected using methods of dynamical systems. Three different flow situations manifest themselves: backward flow, trapping or augmented flow. The transition between backward flow to trapping corresponds to a bifurcation of co-dimension one, in which a non-simple degenerate point changes its stability to form heteroclinic connections between saddle points that enclose recirculating eddies. The transition from trapping to augmented flow is a bifurcation of co-dimension two, in which heteroclinic saddle connections of adjacent waves coalesce below wave troughs. The coalescing of saddle nodes on the longitudinal axis produces a degenerate point with six heteroclinic connections (degenerate saddle). As the parameter is increased, the degenerate saddle bifurcates to saddles nodes which lift off the centerline. These bifurcations are summarized in a global bifurcation diagram. Theoretical results are compared with the experimental data.

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1. Introduction

Peristaltic flow refers to the transport of fluid inside a channel or tube by action of the wall. It is one of the major mechanisms for fluid transport in many biological systems: it is involved in swallowing food through the esophagus, movement of chyme in the gastro-intestinal tract, in the ductus efferentes of the male reproductive system, transport of lymph in the lymphatic vessels and in vasomotion of small blood vessels such as arterioles, venules and capillaries. Peristaltic pumps also work on the same principle, and are used for industrial and medical applications. They can be used to transport corrosive or very pure materials so as to prevent direct contact of the fluid with the pump's internal surfaces.

The dynamics of fluid transport by peristaltic motion of confining walls has received some attention in the literature [1–4]. Early analyses of peristaltic motion were simplified by introducing approximations such as periodic, sinusoidal wave trains in infinitely long tubes or channels, small wall slopes, or low flow Reynolds number. The main objectives were to characterize the basic fluid mechanics of the process and, in particular, to find the pressure gradients that are generated by the wave, the flow behavior in the tube or channel due to peristalsis, and the conditions for trapping or reflux.

Shapiro et al. [1] and Weinberg et al. [5] investigated peristaltic transport by means of an infinite train of peristaltic waves for small Reynolds number and long-wavelength; comparisons between analytical and experimental results were made. Pumping characteristics were investigated for different geometrical parameters. The phenomenon of trapping was studied; trapping refers to the situation where under certain parameter values an internally circulating bolus of fluid, lying about the axis, is transported with the wave speed. Siddiqui and Schwarz [4] analyzed the mechanics of peristaltic pumping for a non-Newtonian fluid through an axisymmetric conduit. A perturbation method in terms of a small wavenumber was applied to solve the problem, and their results showed the presence of trapping under certain conditions. Moreover, it was found that trapping on the longitudinal axis could break up for large positive flow rates; two lifted eddies are maintained below the wave crest in which part of the fluid flows through the center. This behavior was also referred to in the work of Pozrikidis [6], through the imposition of a pressure gradient and using a boundary integral method for Stokes flow.

Peristaltic flow manifests interesting topological changes when parameters are changed, implying that bifurcations can occur. The objective of this work is to exploit methods of dynamical systems to unveil the form and nature of these bifurcations. To describe the

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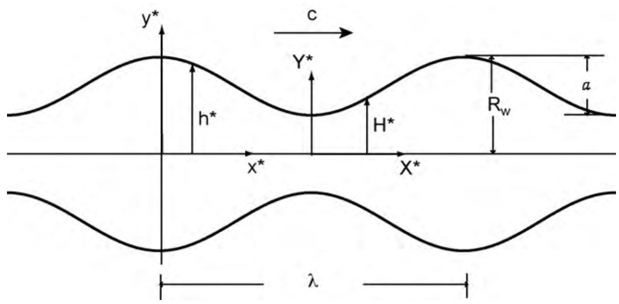

Fig. 1. Geometry of the periodic wave.

fluid dynamics using local properties of streamlines, a Hamiltonian system is considered in which streamlines coincide with particle trajectories. Thus, $\dot{x} = u(x, y) = \partial\psi/\partial y$ and $\dot{y} = v(x, y) = -\partial\psi/\partial x$ where (x, y) are the coordinates, (u, v) are the velocity components and ψ is the stream-function. A stagnation point, where $(u, v) = 0$, is known in dynamics as a critical point, and for two-dimensional incompressible flow there are two non-degenerate possibilities: if the point is a center, the fluid mechanics interpretation is a vortex or eddying motion, while a saddle represents a point of stagnation where the separatrices are the dividing streamlines. This approach to qualitatively describe streamline patterns is by no means new; for basic ideas see for example Hunt et al. [7] and the review paper of Perry and Chong [8]. Further results from this point of view on fluid mechanics can be found in Refs. [9–13]. They used a Taylor expansion of the velocity field and then the coefficients of the series were considered as bifurcation parameters.

Hartnack [10] investigated the streamline patterns and their bifurcations in a two-dimensional incompressible viscous flow near a fixed wall. Normal form theory was used in the local analysis of near wall streamlines. Changes of the streamline patterns near degenerate critical points were investigated. Brøns and Hartnack [11] studied the streamline patterns near simple degenerate points away from boundaries. Topological classifications were obtained in terms of the coefficients of the normal forms. Gürcan and Deliceoğlu [12] studied two-dimensional flows with double symmetry away from boundaries. Classifications of the flow structures were made.

It is the purpose of the present paper to study streamline patterns of two-dimensional peristaltic flow and bifurcations of their critical points. Plane and axisymmetric geometry will be considered. A solution for the stream-function will be presented. Local and global bifurcations of critical points are investigated up to co-dimension two. Theoretical results are compared with the experimental data available in the literature.

2. Plane peristaltic flow

2.1. Flow equations and boundary conditions

Consider peristaltic flow of an incompressible Newtonian fluid in a two-dimensional channel with flexible walls. In a Cartesian coordinate system (X^*, Y^*), the channel walls are shown in Fig. 1, and given by

$$H^*(X^* - ct^*) = R_w - a\left\{1 - \cos^2\left(\pi\frac{X^* - ct^*}{\lambda}\right)\right\},$$

where R_w is total wave height, a is the wave amplitude, λ is the wavelength and c is the wave speed. The motion in a moving frame of reference (x^*, y^*) moving with velocity c of the wave on the walls of the channel is studied. In the wave frame, the motion remains steady. The transformation from a fixed frame to the moving frame are related by $x^* = X^* - ct^*$, $y^* = Y^*$; and the velocity components by $u^* = U^* - c$ and $v^* = V^*$. The dimensional equation of the wall at the moving frame is

$$h^*(x^*) = R_w - a\left\{1 - \cos^2\left(\pi\frac{x^*}{\lambda}\right)\right\}.$$

The dimensional form of the governing equations for the planar problem in the moving frame of reference is

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0,$$

$$v^*\frac{\partial v^*}{\partial y^*} + u^*\frac{\partial v^*}{\partial x^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} + \nu\left[\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial x^{*2}}\right],$$

$$v^*\frac{\partial u^*}{\partial y^*} + u^*\frac{\partial u^*}{\partial x^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu\left[\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial x^{*2}}\right],$$

where ρ is the density, ν is the kinematic viscosity and p^* is the pressure in the moving frame.

Dimensionless quantities are defined as $x \equiv \pi x^*/\lambda$ and $y \equiv y^*/R_w$ for the spatial coordinates. For the velocities $u \equiv u^*/c$ and $v \equiv v^*/\epsilon c$, and for the pressure $p \equiv \epsilon R_w p^*/\mu c$. Two geometrical dimensionless parameters present themselves in this formulation, the amplitude ratio $\phi \equiv a/R_w$, and the wavenumber $\epsilon \equiv \pi R_w/\lambda$. The Reynolds number is defined as $Re \equiv R_w c/\nu$. By introducing the dimensionless stream-function ψ, where $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, which automatically satisfies the continuity equation, and eliminating the pressure from the Navier–Stokes equations by cross-differentiation, the governing equations are reduced to

$$\epsilon Re[\psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y] = \nabla^2(\nabla^2 \psi), \tag{1}$$

where subscripts x and y denote partial differentiation with respect to that variable. The modified Laplacian is given by $\nabla^2 \equiv \epsilon^2 \partial^2/\partial x^2 + \partial^2/\partial y^2$. This is a stream-function formulation of the flow equations.

The dimensionless equation of the wall at the fixed frame is

$$H(X - t) = 1 - \phi\{1 - \cos^2(X - t)\}, \tag{2}$$

where $t \equiv c\pi t^*/\lambda$, $X \equiv \pi X^*/\lambda$, $Y \equiv Y^*/R_w$. The dimensionless equation of the wall in the moving frame is

$$h(x) = 1 - \phi\{1 - \cos^2 x\}. \tag{3}$$

where $0 < \phi < 1$, is the amplitude ratio or occlusion. The dimensional instantaneous flow rate in the fixed frame is given by

$$Q^* = \int_0^{H^*} U^*\,dY^*. \tag{4}$$

Using the transformation between frames for coordinates and velocities, and by integration of Eq. (4) gives

$$Q^* = q^* + ch^*, \tag{5}$$

where $q^* \equiv \int_0^{h^*} u^*\,dy^*$ is the flow rate in the moving coordinate system and is independent of time. The dimensional time-mean flow over a period T at a fixed X-position is defined as

$$\hat{Q} \equiv \frac{1}{T}\int_0^T Q^*\,dt^*. \tag{6}$$

Introducing Eq. (5) into Eq. (6), and using the dimensionless flow rates as $q \equiv q^*/cR_w$ and $Q \equiv \hat{Q}/cR_w$,

$$Q = q + \left(1 - \frac{\phi}{2}\right). \tag{7}$$

Note that the flow rate at the moving coordinate system is also $q = \int_0^h (\partial\psi/\partial y)\,dy = \psi(h) - \psi(0)$. The boundary conditions for this problem are the no-slip condition at the wall, and symmetry conditions along the centerline of the channel. The flow rate relation for the stream-function in the moving frame is selected as zero at