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# Allen-like theory of time for tree-like structures

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ABSTRACT

Allen's Interval Algebra is among the leading formalisms in the area of qualitative temporal reasoning. However, its applications are restricted to linear flows of time. While there is some recent work studying relations between intervals on branching structures, there is no rigorous study of the first-order theory of branching time. In this paper, we approach this problem under a general definition of time structures, namely, tree-like lattices. Allen's work proved that *meets* is expressively complete in the linear case. We also prove that, surprisingly, it remains complete for all unbounded tree-like lattices. This does not generalize to the case of all tree-like lattices, for which we prove that the smallest complete set of relations has cardinality three. We provide in this paper a sound and complete axiomatic system for both the unbounded and general case, in Allen's style, and we classify minimally complete and maximally incomplete sets of relations.



## 1. Introduction

Allen's Interval Algebra (IA [1]) is one of the most prominent formalisms in the area of qualitative temporal, and also spatial, reasoning. However, its applications are naturally restricted to linear flows of time. Recent work focused on integrating intervals and points to obtain a more general first-order language for linear time [2], and to obtain suitable extensions of Allen's IA for branching time with and without points [3,4]. Nonetheless, studying the algebra that emerges from an extended set of relations and its computational complexity is not enough to obtain a clear understanding of the first-order structure that underlies it, and, as witnessed by a famous debate concerning Allen's first-order axiomatization of unbounded interval-based structures (see [5,6]), this study is not always easy. In this work, we provide a throughout analysis of first-order structures for branching time under minimal hypothesis: we only assume that structures are tree-like lattices, and we study both the unbounded and the general case in terms of a suitable pure first-order language to express the relations between two intervals on a tree-like lattice in Ligozat's style [7].

In this work, we present, first, *(i)* a first-order axiomatization of unbounded tree-like lattices, proving that the sole relation *meets* remains complete, and *(ii)* a study of minimally complete and maximally incomplete subsets of linear and branching relations for unbounded tree-like lattices. We then focus on the general case, and we prove that *meets* becomes incomplete when structures are not necessarily unbounded; in this case, we provide, again *(iii)* a first-order axiomatization of tree-like lattices based on a simple complete set of three relations, and *(iv)* a study of minimally complete and maximally incomplete subsets of linear and branching relations for tree-like lattices. This problem was left open in [3] (Re-

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mark 5), and it turned out to be particularly challenging. Apart from the intrinsic interest in generalizing Allen's results to branching models of time, these results give a concrete answer to the question: When is a given abstract relational structure an interval-structure based on a branching model of time? Being able to answer this question is a key step in constraint satisfaction problems over branching models of time. For example, consider a scheduling problem in which there are two processes which are scheduled to start together and after a certain point they will be running in parallel. Each of these two processes may have their own sub-processes with various scheduling requirements. The whole system then is best modeled in a branching model of time and if one wants to check whether a given schedule is consistent one needs to know, among other things, that the abstract relational algebra described by the schedule is indeed based on a branching model of time. Moreover, intervals in branching time are receiving a renewed interested in the field of model checking, historically confined to linear model of time; for example, in [8], the problem of model checking is reformulated in terms of checking interesting properties of computations that make sense on intervals instead of points, and systems give naturally rise to branching structures. Finally, studying the first-order theory of the intervals on branching structures is a necessary preliminary step for the analysis of the complexity of the associated satisfiability (i.e., consistency) problems; these have been partially approached in [3], but are still far from being solved in all aspects.

### *1.1. State of the art*

Various representation theorems exist in the literature for languages that include interval relations only on linear time: van Benthem [9], over rationals and with the interval relations *during* and *before*, Allen and Hayes [10], for the unbounded case without point intervals and for the relation *meets*, Ladkin [11], for point-based structures with a quaternary relation that encodes meeting of two intervals, Venema [12], for structures with the relations *starts* and *finishes*, Goranko, Montanari, and Sciavicco [13], that generalizes the results for structures with *meets* and *met-by*, Bochman [14], for point-interval structures, and Coetzee [15] for dense structure with *overlaps* and *meets*. Branching models of time which can be modeled by tree-like structures are of special interest for temporal reasoning at all levels, since they allow for representing nondeterministic aspects of systems, scenarios, and planning tasks. At the modal logic level branching time models have been studied in depth in the recent past. Originated by philosophical logic [16], where branching time logics have been studied for analysis of non-determinism, causality, and action-theoretical concepts, branching time (point-based) logics such as CTL, CTL*, or ATL (see, e.g. [17,18]) have been proposed as specification languages and, mainly, for model-checking purposes. On the other hand, as far as modal branching time interval-based logics are concerned, the literature is much more scarce; we mention here a future-only branching time version of Propositional Neighborhood Logic studied in [19]. Tree-like structures are a natural choice for modeling temporal aspects of events. For example, in [20] events are defined as closed interval in branching time, and branching structures are exploited to model the different courses that the world might take. The underlying idea is to identify an event with the set of its occurrences in time. An event may occur in many branches (actually an event is said to occur in a branch if and only if it is completely contained in that branch), and a close analysis of this model of time immediately allows one to realize how Allen's relations naturally emerge in this extended context. Also, in the area of automated planning it can be argued that branching time should be chosen as the correct model; as a matter of fact, planning tasks can be modeled as Kripke-like labeled graphs, which naturally unwind to be represented as (potentially unbounded) tree-like structures.

### *1.2. Structure of the paper*

In the next section we give the necessary preliminaries. In Section 3 we prove that *meets* is enough to first-order define any other relation in the context of branching orders, provided that models are unbounded. Section 4, building on the previous result, provides a complete list of minimally complete and maximally incomplete subsets of (linear and branching) relations, again, under the unboundedness assumption. In Section 5 we drop the unboundedness assumption, and show that *meets* is no longer complete; in this context, we provide a complete (and more complex) first-order axiomatization of the set of all branching models. Finally, in Section 6 we study minimal completeness and maximal incompleteness for subsets of (linear and branching) relations in the class of all branching models.

## 2. Preliminaries and notation

Let $(\mathcal{T}, \leq)$ be a partial order, whose elements are generally denoted by $a, b, \ldots, x, y, \ldots$, and where $a||b$ denotes that $a$ and $b$ are incomparable w.r.t. the ordering relation $\leq$. A partial order $(\mathcal{T}, \leq)$, often denoted by $\mathcal{T}$, is a *future branching model of time* if and only if for all $a, b \in \mathcal{T}$ there is a greatest lower bound of $a$ and $b$ in $\mathcal{T}$, and, if $a||b$ then there exists no $c \in \mathcal{T}$ such that $c > a$ and $c > b$. Allen's representation theorem [1] works for the class $\mathsf{Unb}$ of all unbounded (i.e., such that every point has a successor and a predecessor) linearly ordered sets, and it can be immediately generalized for the class $\mathsf{All}$ of all linearly ordered sets [2]. Here, we are interested in: *(i)* the class $\mathsf{TUnb}$ of all unbounded future branching models of time, which immediately generalizes $\mathsf{Unb}$, and *(ii)* the class $\mathsf{TAll}$ of all future branching models of time, which, symmetrically, generalizes $\mathsf{All}$.