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Schnorr randomness for noncomputable measures ☆

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ABSTRACT

This paper explores a novel definition of Schnorr randomness for noncomputable measures. We say x is uniformly Schnorr μ-random if $t(\mu, x) < \infty$ for all lower semicomputable functions $t(\mu, x)$ such that $\mu \mapsto \int t(\mu, x)\, d\mu(x)$ is computable. We prove a number of theorems demonstrating that this is the correct definition which enjoys many of the same properties as Martin-Löf randomness for noncomputable measures. Nonetheless, a number of our proofs significantly differ from the Martin-Löf case, requiring new ideas from computable analysis.

1. Introduction

Algorithmic randomness is a branch of mathematics which gives a rigorous formulation of randomness using computability theory. The first algorithmic randomness notion, Martin-Löf randomness, was formulated by Martin-Löf [1] and has remained the dominant notion in the literature. Schnorr [2], however, felt that Martin-Löf randomness was too strong, and introduced a weaker, more constructive, randomness notion now known as Schnorr randomness. Both Martin-Löf and Schnorr randomness play an important role in computable analysis and computable probability theory. For example, the Martin-Löf randoms are exactly the points of differentiability for all computable functions $f\colon [0,1] \to \mathbb{R}$ of bounded variation [3]. Similarly, the Schnorr randoms are exactly the Lebesgue points for all functions $f\colon [0,1] \to \mathbb{R}$ computable in the L^1-norm [4,5].

Algorithmic randomness is formulated through the idea of "computable tests." Specifically, if μ is a computable measure on a computable metric space $\mathbb{X}$, then (in this paper) a test for Martin-Löf μ-randomness is a lower semicomputable function $t\colon \mathbb{X} \to [0,\infty)$ such that $\int t\, d\mu < \infty$. A point x passes the test t if $t(x) < \infty$, else it fails the test. A point x is Martin-Löf μ-random if x passes all such tests t. Schnorr randomness is the same, except that we also require that $\int t\, d\mu$ is computable for each test t. (We present the full details in the paper.)

While, historically, algorithmic randomness was mostly studied for computable probability measures, there were a few early papers investigating Martin-Löf randomness for arbitrary noncomputable probability measures. One was by Levin [6] using the concept of a "uniform test"—that is, a test t which takes as input a pair (μ, x) and for which $t(\mu, x) = \infty$ if and only if x is μ-random. Gács, later, [7] modified Levin's uniform test approach.[1] Separately, Reimann [8] (also Reimann and Slaman [9]) gave an alternate definition using the concept of a "relativized test"—that is a test t which is computable from (a name for) μ. Day and Miller [10] showed the Levin-Gács and Reimann definitions are equivalent. Recently, there have

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[1] Levin [6] required that uniform tests have two additional properties, called monotonicity and concavity, while Gács [7] removed these conditions. The two approaches lead to different definitions of Martin-Löf μ-randomness for noncomputable measures μ. Gács's approach is now standard.

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been a number of papers investigating Martin-Löf randomness for noncomputable measures, e.g. [11–13,10,14,9]. These results have applications to effective dimension [8], the ergodic decomposition for computable measure-preserving transformations [15], and the members of random closed sets [16,17]—just to name a few.

In stark contrast, Schnorr randomness for noncomputable measures has remained virtually untouched. The first goal of this paper is to give a proper definition of Schnorr randomness for noncomputable measures. Our definition is based on the Levin-Gács uniform tests.

The second goal of this paper is to convince the reader that our definition is the correct one. We will do this by showing that the major theorems concerning Martin-Löf randomness for noncomputable measures also hold for (our definition of) Schnorr randomness for noncomputable measures. While many of the theorems in this paper are known to hold for Martin-Löf randomness, the Schnorr randomness versions require different arguments, using new ideas and tools from computable analysis. However, our proofs naturally extend to Martin-Löf randomness as well. In some cases, we even prove new results about Martin-Löf randomness.

1.1. *Uniform verse nonuniform reasoning*

There are a number of reasons that Schnorr randomness has remained less dominant up to this point. The first is historical: Martin-Löf randomness came first. (Also, much of Schnorr's work, particularly his book [18], was written in German and never translated into English.)

However, there is also another reason: Many consider Schnorr randomness to be less well behaved than Martin-Löf randomness [19, §7.1.2]. Generally two results are given in support of this claim:

1. Schnorr randomness does not have a universal test.
2. Van Lambalgen's Theorem fails for Schnorr randomness.

As for the first point, Martin-Löf showed that there is one universal test t for Martin-Löf randomness such that x is Martin-Löf random iff x passes t. In contrast, for every Schnorr test t there is a computable point (hence not Schnorr random) which fails t. This latter result, while an inconvenience in proofs, actually shows that Schnorr randomness is more constructive. If an a.e. theorem holds for Schnorr randomness (for example, the strong law of large numbers), then we can generally construct a computable pseudo-random object satisfying this a.e. theorem.

As for the second point, Van Lambalgen's Theorem says that a pair (x, y) is Martin-Löf random if and only if x is Martin-Löf random and y is Martin-Löf random relative to x. Whether Van Lambalgen's Theorem holds for Schnorr randomness depends on how one interprets "y is random relative to x." If we use a uniform test approach (similar to the Levin-Gács uniform tests) then it holds [20]. If we use a non-uniform relativized test approach (similar to Reimann's relativized tests) then it does not hold [21]. Uniform approaches were more common in the earlier work of Martin-Löf, Levin, Schnorr, and (to a lesser degree) Van Lambalgen.[2] However, now-a-days it is more common to see nonuniform relativized test approaches. (To be fair, the distinction between uniform and nonuniform reasoning in randomness—and computability theory in general—is quite blurred. This is further exacerbated by the fact that for Martin-Löf randomness, the uniform and nonuniform approaches are equivalent. Nonetheless, one area in computability theory where the distinction between uniform and nonuniform approaches are different is the truth-table degrees and the Turing degrees. Indeed the uniform approach to Schnorr randomness was originally called *truth-table Schnorr randomness* [23].)

This paper is built on the uniform approach, and we believe this goes far in explaining why Schnorr randomness behaves the way it does. Nonetheless, we also briefly look at the nonuniform approach in Subsection 10.3.

1.2. *Finite measures on computable metric spaces*

In this paper, we take a general approach. Instead of working with only Cantor space, we explore randomness for all computable metric spaces. We do this because many of the most interesting applications of randomness occur in other spaces. For example, Brownian motion is best described as a probability measure on the space $\mathcal{C}([0,\infty))$ which is not even a locally compact space. Moreover, the finite-dimensional vector space $\mathbb{R}^d$ is a natural space to do analysis, and any reasonable approach to randomness should be applicable there.

Not only do we consider other spaces, but we also consider finite Borel measures which may not necessarily be probability measures. While, probability theory is mostly concerned with probability measures, other applications of measure theory rely on more general Borel measures. In particular, potential theory (which has had some recent connections with randomness [8,17,24,25]) uses finite Borel measures on $\mathbb{R}^d$.

[2] Indeed, to the extent that Martin-Löf [1, §IV], Levin [6], and Schnorr [18, §24] explored randomness for noncomputable measures—usually Bernoulli measures—their approaches were uniform. One of the two (equivalent) approaches to relative randomness in Van Lambalgen [22, Def. 5.6] is also uniform.