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Effective subsets under homeomorphisms of $\mathbb{R}^n$

Volker Bosserhoff *, Peter Hertling

Fakultät für Informatik, Universität der Bundeswehr München, 85577 Neubiberg, Germany

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ABSTRACT

Which compact subsets of $\mathbb{R}^n$ can be transformed into computable subsets by a homeomorphism of $\mathbb{R}^n$? We show that there exist computably enumerable compact subsets of $\mathbb{R}^n$ and computably coenumerable compact subsets of $\mathbb{R}^n$ that are not mapped to a computable subset by any homeomorphism of $\mathbb{R}^n$.

1. Introduction

In the present article, we demonstrate that even moderate uncomputability of a compact subset of Euclidean space cannot always be “corrected” by a suitable (not necessarily computable) homeomorphism of the space.

For any topological space X, let $\mathcal{C}(X)$ and $\mathcal{K}(X)$ denote the hyperspaces of closed and compact subsets of X, respectively. For any subset $A \subseteq X$, we denote by $\overline{A}$ the closure of A. Let OPENRECT be the set of all nonempty, bounded, open rectangles in $\mathbb{R}^n$ with rational coordinates, and let $I^n : \mathbb{N} \to$ OPENRECT be a canonical total numbering of OPENRECT. If the dimension n is clear from the context we write I instead of I^n.

Definition 1. A set $A \in \mathcal{C}(\mathbb{R}^n)$ is called

- *computably enumerable (c.e.)* if the set

$$\{i \in \mathbb{N} \ : \ I(i) \cap A \neq \emptyset\}$$

  is computably enumerable.

* Corresponding author.
E-mail addresses: bosserhoff@googlemail.com (V. Bosserhoff), peter.hertling@unibw.de (P. Hertling).

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- *computably coenumerable (co-c.e.)* if the set

$$\{i \in \mathbb{N} : \overline{I(i)} \cap A = \emptyset\}$$

  is computably enumerable.
- *computable* if it is both c.e. and co-c.e.

These effectivity notions for subsets of $\mathbb{R}^n$ are natural generalisations of the notions of a computably (co-)enumerable subset of $\mathbb{N}^n$ and a decidable (computable) subset of $\mathbb{N}^n$ to subsets of $\mathbb{R}^n$. They are studied in computable analysis; see [3]. We mention only the fact that a nonempty closed subset A of $\mathbb{R}^n$ is computable if, and only if, its distance function $d_A : \mathbb{R}^n \to \mathbb{R}$ is a computable function.

In this paper we ask for the "topological type" of computable sets. Let us consider two compact subsets $A, B \subseteq \mathbb{R}^n$ as *equivalent* if there is a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(A) = B$. Clearly, this defines an equivalence relation on $\mathcal{K}(X)$. The equivalence class of the empty set contains only the empty set and is uninteresting. But the equivalence class of any nonempty computable compact subset contains also non-computable compact subsets. This raises the question: how complicated does a compact subset of $\mathbb{R}^n$ have to be in order *not* to belong to the equivalence class of any computable subset? Braverman (E-Mail from May 25, 2007) constructed such sets and asked whether there exist co-c.e. or c.e. compact subsets of the plane that are not equivalent to any computable compact subset of the plane. We shall show that this is the case. Denote by Hom_n the group of all homeomorphisms of $\mathbb{R}^n$.

Theorem 2. *Fix some integer $n \geq 1$.*

1. *There is a c.e. set $K \in \mathcal{K}([0,1]^n)$ such that $f(K)$ is not computable for any $f \in \mathrm{Hom}_n$. In case $n \geq 2$, there is such a K that consists only of isolated points and one further connected component.*
2. *There is a co-c.e. set $K \in \mathcal{K}([0,1]^n)$ such that $f(K)$ is not computable for any $f \in \mathrm{Hom}_n$. In case $n \geq 2$, there is such a K that consists only of isolated points and one further connected component.*

This theorem is the main result of the paper. It will be proved in the following sections.

From a computability-theoretic point of view, it may look more natural to consider not arbitrary homeomorphisms but computable homeomorphisms. It turns out that for computable homeomorphisms the situation is much simpler than for arbitrary homeomorphisms. It follows from results by Brattka [1, see p. 327] that the inverse of a computable homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is computable as well. And by [3, Theorem 6.2.4.4], if $A \subseteq \mathbb{R}^n$ is a computable compact set and $f : \mathbb{R}^n \to \mathbb{R}^m$ is computable, then $f(A)$ is a computable compact set as well. Thus, for a compact set $A \subseteq \mathbb{R}^n$ the following two conditions are equivalent:

1. A is computable.
2. There is a computable homeomorphism f of $\mathbb{R}^n$ such that $f(A)$ is computable.

2. Outline of the proof

A key idea of the proof of Theorem 2 is to consider a certain set of natural numbers associated with any closed set.

Definition 3. Suppose $A \in \mathcal{C}(\mathbb{R}^n)$. Let C be the set of isolated points of A, and put $B := A \setminus C$. Define the set $P(A) \subseteq \mathbb{N} \cup \{\infty\}$ by

$$P(A) := \{\mathrm{card}(C \cap U) : U \text{ is a connected component of } \mathbb{R}^n \setminus B\}.$$

The following lemma is obvious.

Lemma 4. *If $A \in \mathcal{C}(\mathbb{R}^n)$ and $f \in \mathrm{Hom}_n$, then $P(A) = P(f(A))$.* □

The basic idea of the proof of Theorem 2 is that for certain "suitable" compact subsets $A \subseteq \mathbb{R}^n$ the set $P(A)$ can be of different computability-theoretic complexity depending on whether A is on the one hand computable or on the other hand c.e. or co-c.e. This suffices since by the previous lemma the set $P(A)$ is invariant under any homeomorphism in Hom_n.

In the following section we give useful characterizations of three classes of sets of natural numbers that will appear in our analysis of the sets $P(A)$ in the various cases to be considered. Then we shall analyze to which class the sets $P(A)$ belong in each case. We will treat the cases $n = 1$ and $n \geq 2$ separately.

We will several times use a standard computable bijective tupling function denoted by $\langle .,. \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Inductively, by $\langle x_1, \ldots, x_n \rangle := \langle \langle x_1, \ldots, x_{n-1} \rangle, x_n \rangle$, we obtain computable bijective tupling functions $\langle ., \ldots, . \rangle : \mathbb{N}^n \to \mathbb{N}$, for $n \geq 3$.