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An abstract approach to stratification in linear logic

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ABSTRACT

We study the notion of stratification, as used in subsystems of linear logic with low complexity bounds on the cut-elimination procedure (the so-called "light" subsystems), from an abstract point of view, introducing a logical system in which stratification is handled by a separate modality. This modality, which is a generalization of the paragraph modality of Girard's light linear logic, arises from a general categorical construction applicable to all models of linear logic. We thus learn that stratification may be formulated independently of exponential modalities; when it is forced to be connected to exponential modalities, it yields interesting complexity properties. In particular, from our analysis stem three alternative reformulations of Baillot and Mazza's linear logic by levels: one geometric, one interactive, and one semantic.

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0. Introduction

The notion of *stratification* in linear logic may be informally presented as a limitation of the dynamics of cut-elimination: in a stratified subsystem of linear logic, proofs may be seen as partitioned into *strata* which never "communicate" with each other, in the sense that no cut between two dual formulas belonging to different strata will ever appear during cut-elimination. All extant time-bounded subsystems of linear logic (with the exception of Lafont's [25] soft linear logic) use some form of stratification to control the complexity of the cut-elimination procedure, which would otherwise be non-elementary (as a consequence of the well known result of [36], modulo the translation of intuitionistic logic in linear logic given by [17]).

In the original systems introduced by [19], namely elementary and light linear logic, stratification coincided with the *exponential depth*, *i.e.*, the nesting level of the logical rules introducing the exponential modality "of course". More recently, [2] introduced a more general form of stratification, still connected with the exponential modalities but no longer coinciding with depth, which keeps ensuring the desired complexity properties.

The present paper originated from a semantic investigation of this more liberal stratification. Our (successful!) attempt to define a denotational semantics for Baillot and Mazza's system naturally revealed that *stratification may actually be formulated independently of exponential modalities; when it is somehow forced to be connected with them, it yields interesting complexity properties*. This is essentially because exponential modalities in linear logic are in control of duplication, the only true source of complexity in cut-elimination.

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The above is the main message brought forth by this paper. We shall now proceed to describe its contents more thoroughly.

0.1. Background

Linear logic, stratification and computational complexity At the heart of our work there is the so-called Curry–Howard correspondence, which sees logical proofs as programs, and cut-elimination as their execution. From this perspective, it is not so much the expressiveness of a logical system *as a language* which matters, but the complexity of its cut-elimination procedure: if a logical system has a low-complexity cut-elimination, its proofs will necessarily correspond to low-complexity programs. This approach, which has a marked proof-theoretic nature and, as such, is orthogonal to the model-theoretic methods of descriptive complexity, falls within the larger area of *implicit computational complexity*, whose concrete aim is to define programming languages enjoying intrinsic complexity bounds, *i.e.*, automatically ensured at compile time. Apart from those already mentioned above, other notable examples of work in this field, not necessarily related to logic, are given by [5,23,22,32].

The use of linear logic as a tool for developing a Curry–Howard-based approach to implicit computational complexity was initiated by [20] and perfected by [19]. The central idea of this latter work is that the complexity of the cut-elimination procedure is mostly owed to the presence of structural rules, in particular the contraction rule. Indeed, the cut-elimination procedure, which is in general non-elementary in the size of proofs [36], becomes quite manageable (*e.g.* quadratic) in substructural logical systems lacking the contraction rule [19]. In linear logic, structural rules are managed by the so-called *exponential* modalities. Girard showed that altering the behavior of these modalities offers a way to define logical systems in which cut-elimination is still feasible (or at most elementary) in spite of the presence of the contraction rule: *light linear logic* (**LLL**) exactly captures deterministic polynomial time, and *elementary linear logic* (**ELL**) exactly captures elementary time.[1]

The restriction that Girard imposed on the exponential modalities of linear logic is a form of *stratification*. Basically, the rules of linear logic are modified so that the nesting level of exponential modalities, called *depth*, may not be changed during cut-elimination. Therefore, a proof may be seen as partitioned into “strata”, one for each depth, which never interact through cut-elimination. We observe that this is not the only use of stratification in implicit computational complexity. For example, [28] introduced *tiers*, which are integers assigned to subterms of λ-terms, to induce a stratification on the λ-calculus, yielding characterizations of interesting complexity classes.

Separating stratification from exponential depth Recently, [2] proposed a new subsystem of linear logic corresponding to elementary time, *linear logic by levels* ($\mathbf{L^3}$). This system is also based on a form of stratification, but in this case it is achieved by retaining only those linear logic proofs π for which there exists a function from the occurrences of formulas in π to the integers, called *indexing*, which satisfies certain conditions. In a nutshell, these conditions state that axioms introduce dual occurrences of identical level, and that the level of an occurrence of formula is decreased only when it is the principal occurrence of a rule introducing an exponential modality.

Interestingly, this form of stratification turns out to be a generalization of Girard's stratification: **ELL** is exactly the subsystem of $\mathbf{L^3}$ in which the function assigning to each occurrence its own depth is a valid indexing. This generalization is strict, both in the sense of proofs and provability: there exist **ELL**-provable formulas which admit more proofs in $\mathbf{L^3}$, and there exist $\mathbf{L^3}$-provable formulas which are not provable in **ELL**. Although no concrete use has currently been found for these additional formulas and proofs, $\mathbf{L^3}$ gives us at least one clear, and potentially interesting message: *stratification does not need to coincide with exponential depth*. However, even if separated from the depth, stratification in $\mathbf{L^3}$ is still explicitly connected to the exponential modalities.

Abstracting stratification through denotational semantics Denotational semantics originated in the work of [34] and [33] as an attempt to interpret in a non-trivial way the quotient induced on λ-terms by β-equivalence. This amounts to finding an invariant of reduction, a question which may be extended to logical systems enjoying cut-elimination. Since its introduction, denotational semantics has proved to be an absolutely essential tool in computer science and proof theory, providing a wealth of information and insights into the nature of computation and formal proofs. A striking example is given by linear logic itself, which arose precisely from a denotational analysis of intuitionistic logic [17].

After the successful introduction of denotational semantics for **LLL**, **ELL** and related systems [3,1,27,10,26], it seemed natural to attempt to analyze the stratification underlying $\mathbf{L^3}$ from the denotational point of view. The result of such an analysis forms the contents of the present paper, whose message broadens that of $\mathbf{L^3}$.

[1] We refer here to the Curry–Howard sense of “capturing”: in these systems, there is a formula F representing functions from binary strings to binary strings such that a proof of F corresponds to a function in the given complexity class and, conversely, every function in that class may be represented by a proof of F.

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