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Common greedy wiring and rewiring heuristics do not guarantee maximum assortative graphs of given degree ☆

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ABSTRACT

We examine two greedy heuristics — wiring and rewiring — for constructing maximum assortative graphs over all simple connected graphs with a target degree sequence. Counterexamples show that natural greedy rewiring heuristics do not necessarily return a maximum assortative graph, even though it is known that the meta-graph of all simple connected graphs with given degree is connected under rewiring. Counterexamples show an elegant greedy graph wiring heuristic from the literature may fail to achieve the target degree sequence or may fail to wire a maximally assortative graph.

1. Introduction

1.1. Motivation

The assortativity of a graph (Newman [1]) is the correlation of the degrees of the endpoints of a randomly selected edge. High degree vertices tend to be connected to high (low) degree vertices in positively (negatively) assortative graphs.

One (of many) practical implications of assortativity is in graph search, e.g., searching a (often large order) graph for (one or all) vertices of maximum (or at least large) degree [2]: the work presented in [3,4] has studied the performance impact of assortativity on search heuristics such as sampling and random walks. Finding such vertices in large graphs has diverse applications, including viral marketing in social networks and network robustness analysis [5,6], among numerous others.

The motivation for this paper is the problem of identifying a collection of graphs, all from the class of graphs with a given degree sequence, with the assortativity of the graphs in the collection varying from the minimum to the maximum possible within that class. The performance impact of the assortativity on the search heuristic may be studied by running the heuristic on all graphs in the collection. Given this objective, the first step is to identify graphs with extremal assortativity within the class. This paper examines two greedy heuristics for finding maximum assortative graphs within a class: graph rewiring and wiring.

1.2. Related work

There is an extensive literature on extremization of assortativity over different graph classes; this section briefly covers the most pertinent points of this literature, focusing on the distinctions between the work presented in this paper and the prior work.

Assortativity. Newman [1] introduced (graph) assortativity which is denoted $\alpha \in [-1, +1]$. Van Mieghem [7] showed perfect assortativity ($\alpha = 1$) is only possible in regular graphs, while any complete bipartite graph $K_{m,n}$

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($m \neq n$) is perfectly disassortative ($\alpha = -1$). There is a large literature on network degree correlations and assortativity (e.g., [8]), and on graphs with extremal assortativity within a class (e.g., [9]).

Joint Degree Matrix (JDM). The generation of random graphs with a particular JDM (also called a 2K-series) has been the subject of a number of recent papers. Stanton [10] and Orsini [8] have proposed random edge rewiring as a method of sampling graphs with a given JDM, while Gjoka [11] has introduced a random wiring method for constructing these graphs. However, there is no means known to us by which JDMs may be efficiently enumerated, and therefore there is no easy means to maximize assortativity, which is a statistic of the JDM, short of enumerating all (in our case, simple and connected) graphs with a given degree sequence.

Rewiring. The meta-graph for a degree sequence, with a vertex for each connected simple graph with that degree sequence and an edge connecting graphs related by rewiring a pair of edges, was studied by Taylor [12]; in particular, he showed this meta-graph to be connected (Thm. 3.3) extending an earlier result by Ryler for simple graphs [13]. This fact is used in §2.

Following Ryler's work, rewiring heuristics for sampling graphs with a particular degree sequence (e.g., [14], [15], [8]) have been introduced. Rewiring heuristics have also been proposed by Newman [16], Xulvi-Brunet [17], Van Mieghem [7], and Winterbach [18] along others for changing a graph's assortativity. The first three of these algorithms, being purely stochastic, cannot efficiently maximize assortativity. Winterbach's algorithm uses a guided rewiring technique to maximize assortativity. However, this technique does not maintain graph connectivity, as its rewirings are a subset of those explored by rewiring heuristic A (see §2.1), and therefore Winterbach's algorithm does not necessarily maximize assortativity.

Wiring. Li and Alderson [19] introduced a greedy wiring heuristic for constructing a graph with maximum assortativity over the set of simple connected graphs with a target degree sequence. Kincaid [9] argues wiring a minimally or maximally assortative connected simple graph is NP-hard and proposes a heuristic which is shown numerically to perform near optimally in minimizing graph assortativity. Winterbach [18], Zhou [20], and Meghanathan [21] have also proposed methods unconstrained by graph connectivity of wiring maximally assortative graphs. This paper examines Li's heuristic further in §3.

Graph enumeration and generation. The results in this paper were achieved using geng, a tool in the nauty package created by McKay [22], to generate all simple connected graphs of a given order.

1.3. Notation

Let $a \equiv b$ denote equal by definition. Let $[n]^+$ denote $\{1, \ldots, n\}$ for $n \in \mathbb{N}$. A graph of order n is denoted $G = (\mathcal{V}, \mathcal{E})$, with vertices $\mathcal{V} = [n]^+$ and edges $\mathcal{E}$; size is denoted by $m = |\mathcal{E}|$. A directed edge between vertices i and j is denoted (ij), and an undirected edge is denoted ij or $\{ij\}$.[1] Let d_i denote the degree of vertex i, $\mathbf{d} = (d_i, i \in \mathcal{V})$ denote a degree sequence, and $\mathbf{d}_G = (d_i, i \in \mathcal{V})$ the degree sequence for graph G. Additionally, let $\mathrm{Uni}(\mathcal{V})$ denote the uniform distribution over vertex set $\mathcal{V}$, $\mathrm{Var}(d_\mathsf{w})$ be the variance of the degree of a randomly selected vertex $\mathsf{w} \sim \mathrm{Uni}(\mathcal{V})$, and $\mathrm{Corr}(d_\mathsf{u}, d_\mathsf{v})$ be the correlation between the degrees of random vertices u and v.

The collection of distinct unlabeled undirected simple connected graphs of order $n \in \mathbb{N}$ is denoted $\mathcal{W}^{(n)}$. Let $\mathscr{D}^{(n)} \equiv \bigcup_{G \in \mathcal{W}^{(n)}} \mathbf{d}_G$ be the collection of degree sequences found in graph collection $\mathcal{W}^{(n)}$, and let $\mathcal{W}^{(n)}_{\mathbf{d}} \equiv \{G \in \mathcal{W}^{(n)} | \mathbf{d}_G = \mathbf{d}\}$ be the graphs in $\mathcal{W}^{(n)}$ with degree sequence $\mathbf{d}$, henceforth referred to as the *degree class* $\mathbf{d}$. It follows that $(\mathcal{W}^{(n)}_{\mathbf{d}}, \mathbf{d} \in \mathscr{D}^{(n)})$ is the partition of $\mathcal{W}^{(n)}$ by the degree sequence $\mathbf{d}$.

The S-metric and assortativity, for $G = (\mathcal{V}, \mathcal{E}) \in \mathcal{W}^{(n)}$, are defined below.

Definition 1. The S-metric [19] is

$$s(G) \equiv \sum_{ij \in \mathcal{E}} d_i d_j. \tag{1}$$

This implies for $\{\mathsf{uv}\} \sim \mathrm{Uni}(\mathcal{E})$ an edge selected uniformly at random $\mathbb{E}[d_\mathsf{u} d_\mathsf{v}] = \frac{1}{|\mathcal{E}|} \sum_{ij \in \mathcal{E}} d_i d_j$. It follows that the assortativity [1] is, for $\mathsf{w} \sim \mathrm{Uni}(\mathcal{V})$ a vertex selected uniformly at random,

$$\alpha(G) \equiv \mathrm{Corr}(d_\mathsf{u}, d_\mathsf{v}) = \frac{s(G)/|\mathcal{E}| - \mathbb{E}[d_\mathsf{w}]^2}{\mathrm{Var}(d_\mathsf{w})}. \tag{2}$$

It is evident that maximizing the S-metric is equivalent to maximizing assortativity over a degree class:

$$\mathcal{W}^{(n)}_{\mathbf{d},\mathrm{opt}} \equiv \operatorname*{argmax}_{G \in \mathcal{W}^{(n)}_{\mathbf{d}}} (s(G)) = \operatorname*{argmax}_{G \in \mathcal{W}^{(n)}_{\mathbf{d}}} (\alpha(G)). \tag{3}$$

Here, $\mathcal{W}^{(n)}_{\mathbf{d},\mathrm{opt}}$ denotes those graphs achieving maximum assortativity over $\mathcal{W}^{(n)}_{\mathbf{d}}$. If there is a unique such graph it is denoted $G^{(n)}_{\mathbf{d},\mathrm{opt}}$.

1.4. Contributions and outline

The rest of the paper is organized as follows. §2 studies several greedy rewiring heuristics, each with the goal of identifying a graph of maximum assortativity over the degree class. Counterexamples are presented showing each of the rewiring heuristics may fail to identify such a graph. §3 examines the greedy wiring heuristic of Li and Alderson [19] designed to identify a graph of maximum assortativity over the degree class. We present a counterexample showing the heuristic may fail to produce a graph in the degree class, and also present a counterexample showing that the heuristic may produce a graph in the class that is not maximally assortative. Both §2 and §3 present tabulations of the number of counterexamples of the various types for

[1] Except in §3 where Algorithm 1's undirected pedges are listed as an ordered pair.