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# Degree sum conditions on two disjoint cycles in graphs ☆

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ABSTRACT

Let $G$ be a simple undirected graph on $n$ vertices. A set of subgraphs of $G$ is *disjoint* if no two of them have any common vertex in $G$. Suppose that $n_1, n_2$ are two integers with $n_1, n_2 \geq 3$ and $n = n_1 + n_2$. We prove that if $d(x) + d(y) \geq n + 4$ for any pair of vertices $x, y$ of $G$ with $xy \notin E(G)$, then $G$ contains two disjoint cycles of length $n_1$ and $n_2$.



## 1. Introduction

We discuss only finite simple graphs and use standard terminology and notation from [3] except as indicated. For any graph $G$, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. The order of $G$ is $|V(G)|$ and the size of $G$ is $e(G) = |E(G)|$. We use $d(u)$ and $\delta(G)$ to denote the degree of $u$ in $G$ and the minimum degree of $G$, respectively. Define

$$\sigma_2(G) = \min\{d(x) + d(y) | x, y \in V(G), xy \notin E(G)\}.$$

A set of subgraphs of $G$ is *vertex disjoint* or simply *disjoint* if no two of them have any common vertex in $G$. The *length* of a cycle $C$ is $|V(C)|$. A *Hamiltonian cycle (path)* of a graph $G$ is a cycle (path) which contains every vertex of $G$. A graph $G$ is *Hamiltonian* if it has a Hamiltonian cycle. A graph $G$ of order $n$ is *pancyclic* if, for every $k$ in the range $3 \leq k \leq n$, $G$ contains a cycle of length $k$.

One of most heavily studied areas in graph theory deals with questions concerning cycles. Lots of results in this area have been given since 1960s. We recall the following classical theorem of Dirac [6].

**Theorem 1.1.** *[6] Let $G$ be a graph. If $|G| = n \geq 3$ and $\delta(G) \geq n/2$, then $G$ is Hamiltonian.*

Ore [16] generalized the theorem and proved the following result.

**Theorem 1.2.** *[16] Let $G$ be a graph. If $|G| = n \geq 3$ and $\sigma_2(G) \geq n$, then $G$ is Hamiltonian.*

Pancyclic graphs are a generalization of Hamiltonian graphs. Bondy [2] extended Ore's theorem by showing that a graph $G$ of order $n$ satisfying $\sigma_2(G) \geq n$ is not only Hamiltonian but even pancyclic, unless $n$ is even and $G$ is isomorphic to a balanced complete bipartite graph $K_{n/2,n/2}$. Other conditions that graphs contain cycles with specified length were also considered [4,5,11,12,14,19,21].

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Let $r$ be a real number. We use $\lceil r \rceil$ for the least integer that is not less than $r$. El-Zahar [7] proved the following result on two disjoint cycles.

**Theorem 1.3.** *[7] Let $n_1$ and $n_2$ be any two integers with $n_1, n_2 \geq 3$. If a graph $G$ of order $n$ with $n = n_1 + n_2$ satisfies $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$, then $G$ has two disjoint cycles of length $n_1$ and $n_2$.*

The degree condition in Theorem 1.3 is best possible, since a complete bipartite graph $K_{n/2,n/2}$ does not have any odd cycle. In the same paper, El-Zahar conjectured that if a graph $G$ of order $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geq 3$ for all $i$ satisfies $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil + \cdots + \lceil \frac{n_k}{2} \rceil$, then $G$ contains $k$ disjoint cycles $C_1, C_2, \ldots, C_k$ of length $n_1, n_2, \ldots, n_k$. Abbasi [1] solved the conjecture for sufficiently large $n$. But the conjecture is still open. The special case $n_1 = n_2 = \cdots = n_k = 4$ was conjectured by Erdős [8] and was proved by Wang [18]. Yan [20] proved that if a graph $G$ of order $n \geq 3s + 4k + 3$ satisfies $\sigma_2(G) \geq n + s$ ($s$ and $k$ are positive integers), then $G$ contains $s$ disjoint triangles and $k$ disjoint quadrilaterals such that all these cycles are disjoint. Gao [10] improved $n \geq 3s + 4k + 3$ to $n \geq 3s + 4k + 1$.

In this paper, we consider the degree sum condition and prove the following theorem, which is also closely related to that of Bondy [2] on the pancyclicity of graphs.

**Theorem 1.4.** *Let $G$ be a graph on $n$ vertices. For any two integers $n_1$ and $n_2$ with $n_1, n_2 \geq 3$ and $n = n_1 + n_2$, if $\sigma_2(G) \geq n + 4$, then $G$ has two disjoint cycles of length $n_1$ and $n_2$.*

The degree condition $\sigma_2(G) \geq n + 4$ in Theorem 1.4 comes from our proof technique. The sharpness of the degree condition of Theorem 1.3 implies that $\sigma_2(G) \geq n + 2$ is necessary for Theorem 1.4. Kostochka and Yu [13] gave a degree sum condition $\sigma_2(G) \geq (4n - 1)/3$ (not sharp in general case) that a graph contains every 2-factor (in particular, a graph has two disjoint cycles each with specified length). This shows that it might not be easy to get a better degree sum condition than one in Theorem 1.4. We ask the following question.

**Question 1.5.** *Let $G$ be a graph of order $n = n_1 + n_2$ with $n_1, n_2 \geq 3$. Can $\sigma_2(G) \geq n + 2$ guarantee that $G$ has two disjoint cycles of length $n_1$ and $n_2$?*

This paper is organized as follows: In Section 2, useful lemmas are given and in Section 3, Theorem 1.4 is proved.

**Notation.** Let $G$ be a simple undirected graph. For two disjoint subgraphs (or vertex subsets) $G_1$ and $G_2$, we define $E(G_1, G_2)$ to be the set of edges of $G$ between $G_1$ and $G_2$, and we write $e(G_1, G_2) = |E(G_1, G_2)|$. If $G_1$ is a single vertex, say $v$, then we simply write $e(v, G_2)$. If $H$ is a subgraph of $G$ and $u \in V(H)$, we use $N_H(u)$ to denote the set of neighbours of $u$ contained in $H$ and $d_H(u) = |N_H(u)|$. We write $N(u)$ for $N_G(u)$ and let $G - H = G[V(G) \setminus V(H)]$ for simplicity. Given a subset $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph of $G$ induced by $U$. For a real number $r$, the biggest integer that is not more than $r$ is denoted by $\lfloor r \rfloor$.

For a cycle $C$ in $G$, we always give $C$ a direction, and we use $C^-$ to denote the cycle of $C$ with its opposite direction. Let $x_j \in V(C)$. We use $x_j^{i-}$ and $x_j^{i+}$ to represent the $i^{th}$ predecessor and successor of $x_j$ on $C$, respectively. Briefly write $x_j^-$ and $x_j^+$ instead of $x_j^{1-}$ and $x_j^{1+}$. We use $C[x_i, x_j]$ to represent the path on $C$ from $x_i$ to $x_j$ along the direction of $C$.

## 2. Lemmas

In order to prove the main theorem, we introduce the following lemmas.

**Lemma 2.1.** *[15] Let $a, b$ be the endvertices of a Hamiltonian path in a graph $G$ of order $n$. If $d(a) + d(b) \geq n$, then $G$ is Hamiltonian.*

**Lemma 2.2.** *[15] Let $C$ be a Hamiltonian cycle of a graph $G$ with order $n$ and let $x$, $y$ be two distinct vertices on $C$. Fix a direction of $C$. If $d_C(x^+) + d_C(y^+) \geq n + 1$ or $d_C(x^-) + d_C(y^-) \geq n + 1$, then $G$ contains a Hamiltonian path with endvertices $x$ and $y$.*

**Lemma 2.3.** *[17] If $P$ is a path of order $k$ in $G$ and $u, v$ are two vertices in $G - V(P)$ such that $e(\{u, v\}, P) \geq k + 2$, then $G[V(P) \cup \{u, v\}]$ has a Hamiltonian path.*

If $G$ is a graph and for any two distinct vertices $u$ and $v$ of $G$, it contains a Hamiltonian path with endvertices $u$ and $v$, then $G$ is *Hamilton-connected*. Erdős and Gallai proved the following result.

**Lemma 2.4.** *[9] If $G$ is a graph on $n$ vertices and $\sigma_2(G) \geq n + 1$, then $G$ is Hamilton-connected.*

We use the following lemma in the proof of the main lemma, Lemma 2.6.

**Lemma 2.5.** *Let $n_1, n_2$ be two integers with $n_1 \geq 5, n_2 \geq 5$ and let $G$ be a graph on $n = n_1 + n_2$ vertices with $\sigma_2(G) \geq n + 4$. Suppose that $(G_1, G_2)$ is a partition of $G$ such that $V(G) = V(G_1) \cup V(G_2)$ and $|G_i| = n_i$ for $i = 1, 2$, and $G_1$ contains a Hamiltonian cycle. For two nonadjacent vertices $x_1, x_2 \in V(G_1)$, if $G_1 - \{x_1, x_2\}$ does not contain a Hamiltonian path, then there exist two vertices $a, b \in V(G_1)$ such that $G_1 - \{a, b\}$ contains a Hamiltonian path and the following holds:*

$$d_{G_1}(a) + d_{G_1}(b) \leq n_1, e(\{a, b\}, G_2) \geq n_2 + 4.$$

**Proof.** Since $\sigma_2(G) \geq n + 4$, for two vertices $u, v$ of $G_1$, the following statement holds:

$$\text{If } uv \notin E(G) \text{ and } d_{G_1}(u) + d_{G_1}(v) \leq n_1, \quad \text{then } e(\{u, v\}, G_2) \geq n_2 + 4. \qquad (*)$$

Suppose that $x_1$ and $x_2$ are two nonadjacent vertices of $G_1$ such that $G_1 - \{x_1, x_2\}$ does not contain a Hamiltonian path. Let $C_1$ be a Hamiltonian cycle of $G_1$. Fix a