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Extremal problems on weak Roman domination number

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ABSTRACT

We show that the weak Roman domination number of a connected n-vertex graph is at most $\frac{2n}{3}$ and characterize the graphs achieving equality. In addition, we provide a constructive characterization of the trees for which the weak Roman domination number equals the Roman domination number and reveal several structural properties of these trees. This answers a problem posed in M. Chellali et al. (2014) [4].

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let $V(G)$ and $E(G)$ be the *vertex set* and *edge set* of a graph G, respectively. For a vertex $v \in V(G)$, we use $d_G(v)$ to denote the *degree* of v in G and let $N_G(v)$ denote the *neighborhood* of v. The *diameter* of G is the maximum distance between vertices of G, denoted by diam(G). An *isolated vertex* is a vertex with degree zero. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. A vertex with at least two leaf neighbors is called a *strong support vertex*. We denote the *star* with one central vertex and k leaves by S_k and the *double star* with exactly two adjacent central support vertices having p and q leaf neighbors by $S_{p,q}$. For two integers i, j such that $i \leq j$, we use $[i, j]$ to denote the set $\{i, i+1, i+2, \ldots, j\}$.

For a graph G, let f be a function from $V(G)$ to $\{0, 1, 2\}$. Denote by $V_i, i = 0, 1, 2$ the set of vertices assigned the label i under f. Thus, f can be viewed as a vertex partition of G such that $V(G) = \{V_0, V_1, V_2\}$, and we can equivalently write $f = (V_0, V_1, V_2)$. We call f an *Roman dominating function* (RDF) of G if every vertex $u \in V_0$ is adjacent to at least one vertex $v \in V_2$. A vertex $u \in V_0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex $v \in V_1 \cup V_2$. We call f a *weak Roman dominating function* (WRDF) of G if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$, such that the function f' defined by $f'(u) = 1$, $f'(v) = f(v) - 1$, and $f'(w) = f(w)$ for all $w \in V(G) \setminus \{u, v\}$ has no undefended vertex. The weight of an RDF (resp. a WRDF) f of G, denoted by $w(f)$, is the value $\sum_{v \in V(G)} f(v)$. The *Roman domination number* $\gamma_R(G)$ (resp. *Weak Roman domination number* $\gamma_r(G)$) is the minimum weight of an RDF (resp. a WRDF) of G. Obviously, $\gamma_r(G) \leq \gamma_R(G)$. An RDF or a WRDF of G with weight ω is called a ω-RDF or ω-WRDF of G. For a subgraph G' of G, we use $f|_{G'}$ to denote the restriction of f to G'.

For a vertex v of a graph G, the *open neighborhood* of v is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \in V(G)$, the *open neighborhood* of S is $N_G(S) = \cup_{v \in S} N_G(v)$ and the *closed neighborhood* of S is $N_G[S] = N_G(S) \cup S$. A vertex u is called a *private neighbor of v with respect to S* or simply an *S-pn* of v if $N_G[u] \cap S = \{v\}$. Note that when $v \notin S$, v has no S-pn. The set pn$(v, S) = N_G[v] - N_G[S \setminus \{v\}]$ of all S-pns of v is called the *private neighbor set of v with respect to S*. The *external private neighbor set of v*, denoted by epn(v, S), is defined as epn$(v, S) = $ pn$(v, S) - \{v\}$. Hence,

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the set epn(v, S) consists of all S-pns of v that belong to $V - S$.

For ease of presentation, we sometimes consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C_T(v)$ and $D_T(v)$ denote the sets of children and descendants of v, respectively. The maximal subtree at v is the subtree of T induced by $D_T(v) \cup \{v\}$ and denoted by T_v.

Roman domination was first studied by Cockayne et al. [5] and has attracted the attention of many scholars; see [3,7,6,9]. Weak Roman domination is a less restrictive version of Roman domination, which was introduced by Henning and Hedetniemi [8]. Regarding the weak Roman domination number, Arumugam [2] and Chellali [4] proposed two problems to ask for a characterization of n-vertex trees T such that $\gamma_r(T) = \frac{2n}{3}$ and $\gamma_r(T) = \gamma_R(T)$, respectively. Recently, José [1] provided a constructive characterization of the trees for which the Roman domination number strongly equals the weak Roman domination number. In this paper, we first show that $\gamma_r(G) \le \frac{2n}{3}$ for a connected n-vertex graph G and characterize the graphs achieving equality, which answers Problem 3.6 proposed by Arumugam et al. [2]. Then, we provide a necessary and sufficient condition to characterize the trees T satisfying $\gamma_R(T) = \gamma_r(T)$ and therefore answer Problem 15 posed by Chellali et al. [4].

2. Weak Roman domination number of connected graphs

This section is devoted to showing that the weak Roman domination number of any n-vertex connected graph is at most $2n/3$ and characterizing the n-vertex graph G satisfying $\gamma_r(G) = 2n/3$. Since adding an edge cannot increase $\gamma_r(G)$, we sufficiently prove the bound for trees.

Theorem 2.1. *If T is an n-vertex tree, $n \ge 2$, then $\gamma_r(T) \le \frac{2n}{3}$.*

Proof. We proceed by induction on n. If $n = 2$ or 3, then T is a path on two vertices or three vertices, and $\gamma_r(T) = 1$ or 2. Let $n \ge 4$. If T is a star, then $\gamma_r(T) = 2 \le \frac{2}{3}n$. If diam$T = 3$, then T is a double star $S_{p,q}$. In this case, when $p \ge 2$ and $q \ge 2$, $\gamma_r(T) = 4$. When $p = 1$ and $q \ge 2$, $\gamma_r(T) = 3$. When $p = 1$ and $q = 1$, $\gamma_r(T) = 2$. Therefore, $\gamma_r(T) \le \frac{2}{3}n$.

In the following, we assume that diam$T \ge 4$ and every n'-vertex tree T' with $n > n'(\ge 2)$ has a WRDF f' with weight at most $\frac{2n'}{3}$. Let $P = u_0u_1 \dots u_m$ be a longest path in T, where $m =$ diam(T). Clearly, $d_T(u_i) \ge 2$ for any $i \in [1, m-1]$, and $d_T(u_0) = d_T(u_m) = 1$.

Case 1. $d_T(u_{m-1}) \ne 3$. Let T' be the subtree of T by removing vertices u_{m-1} and its leaf neighbors. Since diam$T \ge 4$, we have $n' \ge 3$. Define

$$f: V(T) \to \{0,1,2\}: v \mapsto \begin{cases} f'(v), & v \in V(T'), \\ 2, & v = u_{m-1} \text{ and } d_T(u_{m-1}) \ge 4, \\ 1, & v = u_{m-1} \text{ and } d_T(u_{m-1}) = 2, \\ 0, & v \in N_T(u_{m-1}). \end{cases}$$

Then, f is a WRDF of T, while $w(f) = w(f') + 2 \le \frac{2}{3}(n-4) + 2 < \frac{2}{3}n$ (when $d_T(u_{m-1}) \ge 4$) or $w(f) = w(f') + 1 \le \frac{2}{3}(n-2) + 1 < \frac{2}{3}n$ (when $d_T(u_{m-1}) = 2$).

Case 2. $d_T(u_{m-1}) = 3$. Let $N_T(u_{m-2}) \setminus \{u_{m-1}, u_{m-3}\} = \{w_1, w_2, \dots, w_\ell\}$. Since P is the longest path, every vertex in $N_T(w_i) \setminus \{u_{m-2}\}$ is a leaf, and by Case 1, we may assume w_i has either zero or two leaf neighbors. Without loss of generality, let $X_1 = \{w_1, w_2, \dots, w_{\ell'}\}$ and $X_2 = \{w_{\ell'+1}, w_{\ell'+2}, \dots, w_\ell\}$ be the sets of vertices with zero and two leaf neighbors, respectively, where $0 \le \ell' \le \ell$. Here, $X_1 = \emptyset$ when $\ell' = 0$.

Consider the graph by removing edge $u_{m-2}u_{m-3}$ from T; let T_1 be the component containing u_{m-3}, and T_2 be the component containing u_{m-2}. We now define a function f from $V(T)$ to $\{0,1,2\}$ by letting $f(v) = f'(v)$ for any $v \in T_1$, and when $|X_1| \le 1$,

$$f(v) = \begin{cases} 2, & v = u_{m-1} \text{ or } v \in X_2, \\ 1, & v = w_1 \text{ and } |X_1| = 1, \\ 0, & v \in V(T_2) \setminus (X_2 \cup \{u_{m-1}, w_1\}), \end{cases}$$

and when $|X_1| \ge 2$,

$$f(v) = \begin{cases} 2, & v \in (\{u_{m-1}, u_{m-2}\} \cup X_2), \\ 0, & v \in V(T_2) \setminus (X_2 \cup \{u_{m-1}, u_{m-2}\}). \end{cases}$$

Evidently, f is a WRDF of T. When $|X_1| \le 1$, $w(f) = \min\{w(f')+2+1+2(\ell-1)$ (the case of $|X_1| = 1$), $w(f')+2+2\ell$ (the case of $|X_1| = 0$)$\} = \min\{\frac{2}{3}(n-5)+3, \frac{2}{3}(n-4)+2\} < \frac{2}{3}n$. When $|X_1| \ge 2$, $w(f) = f(w')+2+2+2(\ell-\ell_1) \le \frac{2}{3}[n-(3+\ell_1+1+3(\ell-\ell_1))] + 2(\ell-\ell_1) + 4 = \frac{2}{3}n + \frac{4}{3} - \frac{2}{3}\ell_1$. Hence, $w(f) \le \frac{2}{3}n$. In particular, when $|X_1| \ge 3$, we have $w(f) < \frac{2}{3}n$. This also implies that if $w(f) = \frac{2}{3}n$, it must be the case that $d_T(u_{m-1}) = 3$ and $|X_1| = 2$. □

Let f be a $\gamma_R(G)$-RDF (or a $\gamma_r(G)$-WRDF) of a graph G, such that the number of vertices labeled with 1 is the minimum. We call such function f a *special* $\gamma_R(G)$-RDF (or a *special* $\gamma_r(G)$-WRDF) of G and use $\mathscr{F}_R(G)$ (or $\mathscr{F}_r(G)$) to denote the set of all special $\gamma_R(G)$-RDFs (or $\gamma_r(G)$-WRDFs) of G. One can readily check that under a special $\gamma_R(G)$-RDF (or a $\gamma_r(G)$-WRDF), any strong support vertex is labeled with 2. Otherwise, we can assign to the support vertex a weight of 2 and its two leaf neighbors a weight of 0. Particularly, let g be a special $\gamma_R(G)$-RDF of G. Then, G does not contain three vertices u, v, w, such that $vu, vw \in E(G)$ and $f(v) = 0$, $f(u) = f(w) = 1$, or $vu \in E(G)$ and $f(v) = 2$, $f(u) = 1$.

Theorem 2.2. *Let T be a tree on n vertices, $n \ge 3$. Then, $\gamma_r(T) = 2n/3$ if and only if every vertex of degree at least two in T has exactly two leaf neighbors.*

Proof. Let V' and V'' be the set of leaves and vertices with exactly two leaf neighbors in T, respectively. If $V'' = V \setminus V'$, then $n = 3|V''|$. Let $f \in \mathscr{F}_r(T)$. Then, every vertex in V'' is labeled with 2, and every vertex in V' is labeled with 0 under f. Therefore, $\gamma_r(T) = w(f) = 2|V''| = 2n/3$.

Conversely, if $\gamma_r(T) = 2n/3$, then let g be a special $\frac{2n}{3}$-WRDF of T. Let $P = u_0u_1 \dots u_m$ be the longest path