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On fan-crossing and fan-crossing free graphs

Franz J. Brandenburg

94030 Passau, Germany

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ABSTRACT

A drawing of a graph in the plane is *1-planar* if each edge is crossed at most once, *fan-crossing* if each edge can only cross edges with a common endpoint, and *fan-crossing free* if no edge is crossed by two or more edges with a common endpoint. In an *outer* drawing all vertices must be placed in the outer face. A graph is (outer-) 1-planar (fan-crossing, fan-crossing free) if it admits a respective drawing.

Fan-crossing and fan-crossing free are complementary properties. Hence, a drawing is 1-planar if and only if it is fan-crossing and fan-crossing free. We show that there are graphs that are simultaneously (outer-) fan-crossing and fan-crossing free but not (outer-) 1-planar.

1. Introduction

Graphs with or without special patterns for edge crossings in drawings are an important topic in Topological Graph Theory, Graph Drawing, and Computational Geometry. Particular patterns are no crossings, single crossings, fan crossings, and crossings by independent edges. A *fan* is a set of at least two edges with a common endpoint. In complement, two or more edges are *independent* if they do not share a common endpoint, see Fig. 1. A drawing (or embedding) of a graph is called *fan-crossing* if each edge is uncrossed, crossed at most once or crossed by a fan, and *fan-crossing free* if there are no fan-crossings. Then there may be independent crossings. Fan-crossings are also known as radial $(k, 1)$-grid crossings [26] and independent crossings as natural grids [1]. A graph is called *fan-crossing* (*fan-crossing free*) if it admits a respective drawing. A graph is *1-planar* if it can be drawn so that each edge is crossed at most once. A first order logic definition of these and other graph classes is given in [12]. Obviously, every drawing that is simultaneously fan-crossing and fan-crossing free is 1-planar, and conversely. Hence, every 1-planar graph is both fan-crossing and fan-crossing free so that the class of 1-planar graphs is included in the intersection of the classes of fan-crossing and fan-crossing free graphs.

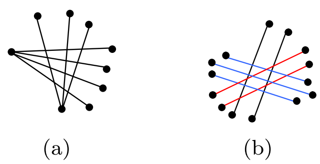

Fig. 1. (a) Fan-crossings and (b) independent-crossings or fan-crossing free.

Important subclasses of graphs were introduced by placing all vertices in the outer face of a drawing, or equivalently, in (strictly) convex position. A graph is *outerplanar* if it can be drawn in the plane so that all vertices are in the outer face and edges do not cross. *Outer-1-planar* (*outer-fan-crossing*, *outer-fan-crossing* free) graphs are defined accordingly.

(Outer-) 1-planar graphs and graphs with or without fan-crossings have recently found interest and have been studied in several works with emphasis on the density of graphs, the complexity of the recognition problem, or the relationship to other classes of graphs, see e.g.,

E-mail address: brandenb@informatik.uni-passau.de.

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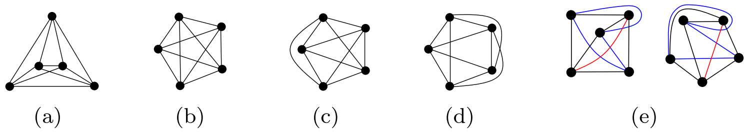

Fig. 2. All non-isomorphic embeddings of K_5. Only (a) is 1-planar and fan-crossing free, (b), (c), and (d) are fan-planar and (e) (shown in two drawings) has no crossing of independent edges but an edge, drawn red, that is crossed by the edges of a triangle. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

[12,18–20,25]. Inclusion relationships show, which properties of graphs in one class can be carried over to the graphs in another class, and an incomparability indicates different properties.

Kaufmann and Ueckerdt [24] introduced fan-planar graphs which exclude independent crossings and fan-crossings in which an edge is crossed by edges of a fan from both sides. They proved that n-vertex fan-planar graphs have at most $5n - 10$ edges, and that this bound is tight. The study of fan-planar graphs was continued in several works. It is known that the recognition problem of fan-crossing graphs is NP-hard [5,7]. The author [10] recently showed that the exclusion of fan-crossings from both sides is essential but has no impact on the maximal density of graphs with at most $5n - 10$ edges, and that the 8-clique minus one edge, that is $K_8 - e$, is not a fan-crossing graph.

Fan-crossing free graphs were studied by Cheong et al. [15], who showed that they have at most $4n - 8$ edges and that every fan-crossing free graph with $4n - 8$ edges consists of a planar quadrangulation and a pair of crossing edges in each face. These are extensions of similar results on 1-planar graphs [8,27]. The density of $4n - 8$ for 1-planar graphs has been proved at several places. Graphs with the maximum number of edges are called *optimal*. Hence, the classes of optimal 1-planar graphs and optimal fan-crossing free graphs coincide. Thus optimal fan-crossing free graphs can be recognized in linear time [9]. We are not aware of a paper with a proof of the NP-hardness of the recognition problem of fan-crossing free graphs. A proof can be obtained by a minor modification of the proof by Grigoriev and Bodlaender [21], where the edges in the circles of the double wheels are replaced by triangles.

Outer-1-planar graphs with n vertices have at most $2.5n - 4$ edges [2,16]. They are planar and have book thickness two and treewidth three and can be recognized in linear time [2,23]. Maximal outer-fan-planar graphs can be recognized in linear time [5]. Outer fan-crossing free graphs have not yet been studied. More general outer graph classes were studied in [14].

In this paper it is shown that there are graphs that admit two embeddings, one is fan-crossing and the other is fan-crossing free, and neither of them is 1-planar. In fact, there are graphs that are fan-crossing and fan-crossing free but not 1-planar. Moreover, there are graphs that are outer-fan-crossing and outer-fan-crossing free but not outer-1-planar. In the remainder of this paper, we introduce our notation and establish properties of 1-planar embeddings in Section 2, prove our main result in Section 3 and conclude with some open problems in Section 4.

2. Preliminaries

We consider undirected graphs $G = (V, E)$ with finite sets of vertices V and edges E that are *simple* both in a graph theoretic and in a topological sense. Thus we do not allow multiple edges and self-loops, and we exclude multiple crossings of two edges and crossings among adjacent edges.

A *drawing* of a graph G is a mapping of G into the plane so that the vertices are mapped to distinct points and each edge is mapped to a Jordan arc between the endpoints. Two edges *cross* if their Jordan arcs intersect in a point other than an endpoint. A drawn graph is called a *topological graph*. In other works, a topological graph is called an *embedding* which is the class of topologically equivalent drawings. Its picture is a *planarization* in which each crossing point is treated as a vertex. A drawn graph partitions the plane into topologically connected regions, called *faces*. The unbounded region is called the *outer face*. The *boundary* of each face consists of a cyclic sequence of segments of edges and is commonly specified by a sequence of vertices and crossing points.

The subgraph of a graph G induced by a subset U of vertices is denoted by $G[U]$. It inherits its embedding from an embedding of G, from which all vertices not in U and all edges with at most one endpoint in U are removed. For an edge e, let $G - e$ denote the subgraph obtained from G by removing e.

It may be hard to prove that a graph is not 1-planar. This is apparent by the NP-hardness of the recognition problem [21], even if the graphs are 3-connected and are given with a rotation system [3]. Also, forbidden minors cannot help, since the 1-planar graphs are not minor closed. Clearly, a graph is not 1-planar, if it has a subgraph with too many edges, namely at least $4k - 7$ for a k-vertex subgraph.

In this work, we use the uniqueness of 1-planar embeddings of K_5 up to isomorphism as a tool to prove that certain graphs are not 1-planar. The aggregation of several K_5 leads to a subgraph that has a unique embedding when restricted to a distinguished spanning subgraph whose edges must be embedded uncrossed.

The 5-clique K_5 has five embeddings, as shown by Harborth and Mengersen [22]. They are shown in Fig. 2. Only the one in Fig. 2(a) is 1-planar. This embedding, called *pyramid embedding*, is distinguished by a top vertex t so that all edges incident to t are uncrossed, and by four triangles, whose boundary consists of vertices and does not contain a crossing point. One of these faces is used as the outer face.