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Notes on complexity of packing coloring

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ABSTRACT

A packing k-coloring for some integer k of a graph $G = (V, E)$ is a mapping $\varphi : V \to \{1, \dots, k\}$ such that any two vertices u, v of color $\varphi(u) = \varphi(v)$ are in distance at least $\varphi(u) + 1$. This concept is motivated by frequency assignment problems. The *packing chromatic number* of G is the smallest k such that there exists a packing k-coloring of G.

Fiala and Golovach showed that determining the packing chromatic number for chordal graphs is NP-complete for diameter exactly 5. While the problem is easy to solve for diameter 2, we show NP-completeness for any diameter at least 3. Our reduction also shows that the packing chromatic number is hard to approximate within $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$.

In addition, we design an FPT algorithm for interval graphs of bounded diameter. This leads us to exploring the problem of finding a partial coloring that maximizes the number of colored vertices.

1. Introduction

Given a graph $G = (V, E)$ and an integer k, a *packing k-coloring* is a mapping $\varphi : V \to \{1, \dots, k\}$ such that any two vertices u, v of color $\varphi(u) = \varphi(v)$ are in distance at least $\varphi(u) + 1$. An equivalent way of defining the packing k-coloring of G is that it is a partition of V into sets $V_1, \dots, V_k$ such that for all k and any $u, v \in V_k$, the distance between u and v is at least $k + 1$. The *packing chromatic number* of G, denoted $\chi_P(G)$, is the smallest k such that there exists a packing k-coloring of G.

The definition of packing k-coloring is motivated by frequency assignment problems. It emphasizes the fact that the signal on different frequencies can travel different distances. In particular, lower frequencies, modeled by higher colors, travel further so they may be used less often than higher frequencies. The packing coloring problem was introduced by Goddard et al. [11] under the name *broadcasting chromatic number*. The term packing coloring was introduced by Brešar, Klavžar, and Rall [2].

Determining the packing chromatic number is often difficult. For example, Sloper [14] showed that the packing chromatic number of the infinite 3-regular tree is 7 but the infinite 4-regular tree does not admit any packing coloring by a finite number of colors. Results of Brešar, Klavžar, and Rall [2] and Fiala, Klavžar and Lidický [8] imply that the packing chromatic number of the infinite hexagonal lattice is 7.

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Looking at these examples, researchers asked the question if there exists a constant p such that every subcubic graph has packing chromatic number bounded by p. A very recent result of Balogh, Kostochka and Liu [1] shows that there is no such p in quite a strong sense. They show that for every fixed k and $g \geq 2k+2$, almost every n-vertex cubic graph of girth at least g has packing chromatic number greater than k. It is still open if a constant bound holds for planar subcubic graphs, and no deterministic construction of subcubic graphs with arbitrarily high packing chromatic number is known.

Despite a lot of effort [8,11,13,15], the packing chromatic number of the square grid is still not determined. It is known to be between 13 and 15 due to Barnaby, Franco, Taolue, and Jos [13], who use state of the art SAT-solvers to tackle the problem. In this paper, we consider the packing coloring problem from the computational complexity point of view. In particular, we study the following problem.

Packing k-coloring of a graph	
Input:	A graph G and a positive integer k.
Question:	Does G allow a packing k-coloring?

1.1. Known results

We characterize our algorithmic parameterized results in terms of FPT (running time $f(k)\text{poly}(n)$) and XP (running time $n^{f(k)}$) where n is the size of the input, k is the parameter and f is any computable function. The investigation of computational complexity of packing coloring was started by Goddard et al. [11] in 2008. They showed that packing k-coloring is NP-complete for general graphs and $k=4$ and it is polynomial time solvable for $k \leq 3$. Fiala and Golovach [7] showed that packing k-coloring is NP-complete for trees for large k (dependent on the number of vertices).

For a fixed k, packing k-coloring is expressible in MSO_1 logic. Thus, due to Courcelle's theorem [4], it admits a fixed parameter tractable (FPT) algorithm parameterized by the tree-width or clique-width [5] of the graph. Moreover, it is solvable in polynomial time if both the tree-width and the diameter are bounded [7]. The problem remains in FPT even if we fix the number of colors that can be used more than once by the extended framework of Courcelle, Makowsky and Rotics [5], see Theorem 11. On the other hand, the problem is NP-complete for chordal graphs of diameter exactly 5 [7], and it is polynomial time solvable for split graphs [11]. Note that split graphs are chordal and have diameter at most 3. However, packing k-coloring admits an FPT algorithm on chordal graphs parameterized by k [7].

1.2. Our results and structure of the paper

In Section 2, we describe new complexity results on chordal, interval and proper interval graphs. We improve a result by Fiala and Golovach [7] in Theorem 5, where we show that computing packing chromatic number of chordal graphs of any diameter greater or equal than three is NP-complete. Moreover, the reduction implies an inapproximability result based on the inapproximability of the size of the largest independent set. Proposition 3 shows that calculating the packing chromatic number of chordal graphs of diameter less than three is can be done in polynomial time.

We complement these results by several FPT and XP algorithms for calculating the packing chromatic number on interval and proper interval graphs. We use dynamic programming to get an XP algorithm for interval graphs of bounded diameter, see Theorem 6. For unit interval graphs, there is an FPT algorithm parameterized by the size of the largest clique, see Theorem 9. Note that the existence of an FPT algorithm for calculating the packing chromatic number parameterized by path-width would imply an FPT algorithm for general interval graphs parameterized by the size of the largest clique, but existence of such algorithm remains an open question. We also provide an XP algorithm calculating the packing chromatic number for interval graphs parameterized by the number of colors that can be used more than once, see Theorem 10.

In Subsection 2.1, we describe complexity results and algorithms parameterized by structural parameters. We design FPT algorithms for them. For standard notation and terminology we refer to the recent book about parameterized complexity [6].

The packing coloring problem is interesting only when the number of colors is not bounded. Otherwise, we can easily model the problem by a fixed MSO_1 formula and use the FPT algorithm by Courcelle [4] parameterized by the clique-width of the graph. We show that we can do a similar modeling even when we fix only the number of colors that can be used more than once and then use a stronger result by Courcelle, Makowsky and Rotics [5] that gives an FPT algorithm parameterized by clique-width of the graph (Theorem 11).

If the number of such colors is part of the input, then we can solve the problem on several graph classes. If they have a bounded diameter, then we can use Theorem 11 due to the following easy observation.

Observation 1. Let G be a graph of bounded diameter. Then G has a bounded number of colors that can be used more than once.

This observation together with Theorem 11 implies that the problem is FPT for any class of graphs of bounded shrub-depth. Any class of graphs that has bounded shrub-depth has a bounded length of induced paths ([10], Theorem 3.7) and thus bounded diameter. The same holds for graphs of bounded modular-width as they have bounded diameter according to Observation 2. On the other hand, the problem was shown to be hard on graphs of bounded tree-width [7], in fact the problem is NP-hard even on trees. There seems to be a big gap and thus interesting question about parameterized complexity with respect to path-width of the graph. It still remains open (Question 14). Note that the original hardness reduction by Fiala and Golovach [7] has unbounded path-width. See Fig. 2 for an overview of the results with respect to the structural