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3-coloring arrangements of line segments with 4 slopes is hard

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ABSTRACT

In a paper first appeared at SODA '07, Eppstein proved that testing the 3-colorability of arrangements of line segments is an *NP*-complete problem. However, if the slopes of the segments are limited to three different values, a 3-coloring can be trivially obtained by assigning the same color to all the segments having the same slope.
We thus study the complexity of testing the 3-colorability of arrangements of line segments, or equivalently of their intersection graphs, that are restricted to have a constant number $s > 3$ of slopes, and prove that this remains *NP*-complete even for $s = 4$, which is hence tight. More in general, we prove that k-coloring arrangements of line segments is *NP*-complete even if the segments have at most $k + 1$ slopes.
Since the problem of computing a k-coloring of an arrangement of line segments is equivalent to computing the constrained geometric thickness of a straight-line drawing of a graph in the plane, our result extends to this problem.

1. Introduction

Coloring geometric objects in the plane in such a way that no two intersecting objects are assigned the same color is a famous problem in Graph Theory [1–5], bringing together the two well-known problems of graph coloring [6] and of intersection representations of graphs [7].

In fact, given a set of objects in the plane, one can consider each object as a vertex of a graph whose edges are defined by the intersection pattern of the set of objects. This corresponds to interpreting the arrangement of the objects as a geometric intersection representation of the graph. Under this interpretation, coloring the objects in such a way that intersecting objects have different colors is equivalent to coloring the vertices of the graph so that adjacent vertices have different colors.

On the other hand, when the objects in the plane are (straight-)line segments, each of such objects may also be interpreted as an edge whose endvertices are placed at its endpoints. In this case, the whole representation would comprise a (straight-line) drawing of the resulting graph, and the problem of coloring the objects would correspond to the problem of computing a *planar stratification* [8] of such drawing, that is, an edge-coloring such that each set of edges having the same color induces a planar drawing. The minimum number of colors needed for obtaining a planar stratification of a straight-line drawing is also known as the *constrained geometric thickness* [3,9] of such drawing.

The problem of coloring objects in the plane using a fixed number k of colors has been proved *NP*-complete with $k \geq 3$ for several types of objects, including simple ones like unit disks [2] and straight-line segments [3]. On the other hand, solving the problem for $k = 2$ can be done in polynomial time, as it corresponds to determining whether the intersection graph describing the intersection pattern is bipartite. In particular, an $O(n \log n)$-time algorithm exists for some types of geometric objects when

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$k = 2$, e.g., straight-line segments and simple polygons in the plane [3].

In this paper we consider the setting of the problem in which the objects that have to be k-colored are straight-line segments, which we call LINE-SEGMENT k-COLORABILITY PROBLEM (k-LSC). In particular, we study the complexity of k-LSC when the number s of different slopes of the segments is bounded. The interest in this restriction comes from the observation that, if s is chosen to be small enough (in particular, $s \leq k$), then k-LSC becomes trivial. In fact, since parallel segments cannot intersect each other,[1] assigning the same color to all the segments with the same slope always yields a valid solution. We prove that k-LSC remains *NP*-complete with $k \geq 3$ even if $s = k + 1$. Due to the above observation, this result is tight with respect to the number of slopes. We remark that the reduction provided by Eppstein [3] to prove *NP*-completeness of k-LSC with $k \geq 3$ may produce instances in which $s \in \Omega(n)$.

The paper is structured as follows. In Section 2 we give basic definitions and present some related work. In Section 3 we show our main result. Finally, in Section 4 we give concluding remarks and discuss some open problems.

2. Preliminaries

A graph $G = (V, E)$ is a pair composed of a set V of vertices and a set $E \subseteq V^2$ of edges. A graph without self-loops and multi-edges is *simple*. In the following, we will only consider simple graphs. A *matching* is a graph composed of independent edges.

A *drawing* of a graph is a mapping of each vertex to a point of the plane and of each edge to a simple curve connecting its endpoints. A drawing of a graph is *planar* if the curves representing the edges do not cross except, possibly, at common endpoints. A graph is *planar* if it admits a planar drawing.

The *slope* of a segment is the angular coefficient of the line passing through it. Given a set L of line segments, we denote by $s(L)$ the number of different slopes of the segments in L.

A *visibility representation* of a planar graph $G = (V, E)$ maps each vertex in V to a horizontal segment and each edge (u, v) in E to a vertical segment not intersecting any horizontal segment and whose endpoints lie on the horizontal segments representing u and v. Visibility representations can be constructed in polynomial time [10,11].

Given a planar graph G, the PLANAR GRAPH k-COLORABILITY PROBLEM (k-PGC) asks whether there exists a *k-coloring* of G, that is, a coloring of the vertices of G with at most k different colors so that no two adjacent vertices have the same color. Given a set L of open line segments in the plane such that any two of them intersect in at most one point, the LINE-SEGMENT k-COLORABILITY PROBLEM (k-LSC) asks whether there exists a *k-coloring* of L, that is, a coloring of the segments in L with at most k different colors so that no two intersecting segments have the same color.

Given a graph G and a straight-line drawing Γ of G, the PLANAR k-STRATIFICATION PROBLEM (k-PS) asks whether it is possible to partition the edges of G into at most k different sets in such a way that no two edges in the same set intersect in Γ. Clearly, if we consider the set L that contains an open line segment for each edge of G in Γ, then a bijection exists between the k-colorings of L and the valid partitions of the edges of G in Γ that determine a solution of k-PS. Observe that, given a straight-line drawing Γ of G, the minimum k such that $\langle G, \Gamma \rangle$ is a positive instance of k-PS coincides with the constrained geometric thickness of Γ.

The k-LSC problem is known to be *NP*-complete [3] for $k \geq 3$. The equivalence between k-LSC and k-PS implies the *NP*-completeness of k-PS as well, even though this connection has gone unnoticed in the original paper defining the k-PS problem [8].

3. Reduction

In this section we describe the main result of the paper.

Theorem 1. *The* LINE-SEGMENT k-COLORABILITY PROBLEM *of computing a k-coloring of a set L of line segments is NP-complete for $k \geq 3$ even if $s(L) = k + 1$.*

Proof. The membership in *NP* follows from the membership of the general problem in which there is no restriction on the value of $s(L)$ [3]. We first prove the *NP*-hardness for $k = 3$ by means of a polynomial-time reduction from the *NP*-complete problem 3-PGC [12]. We defer the extension to $k > 3$ to the end of the proof. We note that the *NP*-hardness of the general problem was also proved [3] by means of a reduction from 3-PGC. However, in our reduction we exploit the properties of visibility representations to obtain an instance of k-LSC with a bounded number of slopes.

Given a planar graph G, we construct a set L of line segments with $s(L) = 4$ such that G admits a 3-coloring if and only if L admits a 3-coloring.

As a first step, construct a visibility representation Γ of G on the integer grid by using any of the known polynomial-time algorithms [10,11]. We assume that in Γ any two vertical segments have horizontal distance at least 4 and any two horizontal segments have vertical distance at least 5. This is without loss of generality as new rows and columns can be introduced while preserving the representation.

We consider the edges of G as oriented downward according to Γ. Namely, whenever we refer to edge $e = (u, v)$ of G we assume that the horizontal segment representing u in Γ lies above the horizontal segment representing v in Γ.

Refer to Fig. 1. For each horizontal segment representing a vertex u of G in Γ, add to L a copy $s_0(u)$ of such a segment. Then, for each edge $e = (u, v)$ in G, consider the vertical segment representing e in Γ. Let a and b be the endpoints of such a segment, where $y(a) > y(b)$. Add to L a vertical segment $s_\infty(e)$ with endpoints a' and

[1] Degenerate intersections are usually not allowed in this context, that is, any two segments share at most one point and segments are considered as open curves. The latter restriction implies that collinear segments sharing an endpoint are not considered to be crossing.