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Linkage on the infinite grid

Adam S. Jobson [a], André E. Kézdy [a], Jenő Lehel [a,b,*]

[a] University of Louisville, Louisville, KY 40292, United States of America
[b] Alfréd Rényi Mathematical Institute, Budapest, Hungary

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For k fixed, a graph G is *k-path-pairable*, if for any set of k disjoint pairs of vertices, s_i, t_i, $1 \le i \le k$, there exist pairwise edge-disjoint s_i, t_i-paths in G. Bounds on path-pairability are given here if G is the graph of the infinite integer grid in the Euclidean plane (vertices of G are the points of integer coordinates and two vertices are adjacent if and only if their Manhattan distance is 1). We prove that G is 10-path-pairable and at most 14-path-pairable. Related results and conjectures are summarized also for the integer halfplane, for the positive integer quadrant and for finite grids.

1. Introduction

The board of the game is an arbitrary (even infinitely) large grid drawn on a graph paper. An integer label from $1, 2, \ldots, k$ is placed at some crossings, each integer occurring at exactly two places. The goal is to link the k pairs of the identically labeled points by paths along the lines of the board in such a way that those paths do not share a common line (although they may share common points).

Let's have a round with $k = 4$ pairs placed randomly on a 5×5 board! Can you succeed on the boards in Fig. 1? Can you devise a practical procedure which solves the problem for any placement of those eight integer labels, say on a 19×19 board?

The different variations of our 'linkage solitaire' are not unknown subjects in discrete mathematics. As a basic framework one should mention the theory of multicommodity flows brilliantly surveyed in [16]. The restriction of finding integer multiflows under unit capacity constraints leads to different concepts in pure graph theory. One must cite here fundamental edge-connectivity results due to Menger [12], the investigations pertaining to disjoint paths in [15], and further results on different linkage properties (e.g. in [3], [4], [6], [13], [14]).

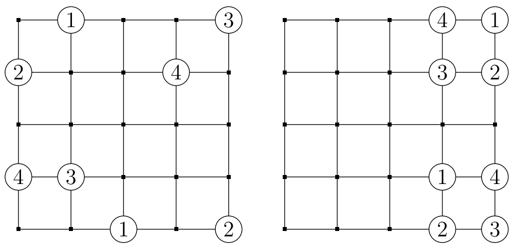

Fig. 1. Two boards to check 4-path-pairability.

Finding disjoint paths in grids has applications in many practical problems. The point-to-point delivery problem (see [11]) is to determine a set of disjoint 'shipping' paths matching the sources to destinations. A recent application where linkage problems on 2-dimensional grids are typical is VLSI-design (see [1]). In very large-scale integrated circuits several pairs of pins must be inter-connected by

* Corresponding author.
E-mail addresses: adam.jobson@louisville.edu (A.S. Jobson), andre.kezdy@louisville.edu (A.E. Kézdy), j0lehe01@louisville.edu (J. Lehel).

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wires on a chip, in such a way that the wires follow given 'channels' and that the wires connecting different pairs of pins do not intersect each other.

The linkage game treated here and the general concept of path-pairability originated in a practical problem concerning telecommunications network as first described in [2]. Such a data- or telephone network is a collection of terminals (hosts), links, and intermediate nodes which are assembled so as to enable communication between the terminals. In typical applications, pairs of communicating terminals are connected through transmission links. Suppose that a terminal S wishes to communicate with another terminal T. In circuit switching mode, a dedicated communication link is allocated between S and T, via a set of intermediate nodes and this linkage or path is maintained during the communication between S and T. It is assumed that no two communication paths between distinct pairs can share common communication lines, although the paths can use common intermediate nodes. We require the network to allow messages to be passed simultaneously between any fixed number of disjoint pairs of nodes of the network.

Modeling the practical telecommunications networks problem led to the concept of path-pairability of graphs in Csaba et al. [2]. For fixed k, a graph G is *k-path-pairable*, if for any set of k disjoint pairs of vertices called *terminals*, s_i, t_i, $1 \le i \le k$, there exist pairwise edge-disjoint s_i, t_i-paths in G. The *path-pairability number*, denoted $pp(G)$, is the largest k such that G is k-path-pairable.

2. Path-pairability of the infinite grid

Let G be the graph of the infinite integer grid in the Euclidean plane $\mathbb{R}^2$. The vertices of G are the points of $\mathbb{R}^2$ with integer coordinates and the points (x_1, y_1) and (x_2, y_2) are adjacent vertices of G if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. Thus the edges of G are arranged into vertical and horizontal lines of the plane (into two-way infinite paths – in terms of graph theory).

When proving that G is not k-path-pairable, we must exhibit a counterexample consisting of k terminal pairs that does not admit a linkage for the pairs. To see that no solution exists we use a necessary condition, called *cut condition*, and/or its ramifications.

As an example, the bound $pp(G) < 19$ follows from a pairing where one terminal from each of 19 pairs is located at distinct points of a 4×5 rectangle $R \subset G$ and the second members of the pairs are located anywhere in $G - R$. If there is a linkage for the pairs, then 19 edge-disjoint paths must 'escape' from R, but there are only 18 outlets leading from R to $G - R$. In other words, as few as 18 edges cut R from $G - R$, thus we might say, there is no solution because the cut condition is violated by this particular pairing. Sharper upper bounds can be obtained by using stronger necessary conditions derived from the cut condition as in the next proposition.

Proposition 1. *The infinite grid is not* 15*-path-pairable.*

Proof. Let $Q \subset G$ be a 5×5 square grid with its four corners removed (see Fig. 2). We place terminals at all vertices of Q such that the terminals on the boundary form the pairs $\pi_i = \{s_i, t_i\}$, $1 \le i \le 6$, and the pair of each terminal at an interior vertex of Q is placed anywhere in $G - Q$. Assume on the contrary that there is a linkage for these 15 pairs.

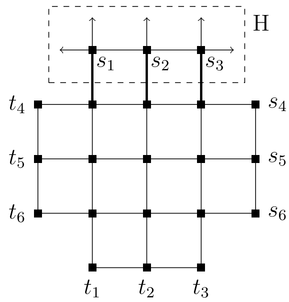

Fig. 2. $pp(G) < 15$.

There are 21 terminals located at the vertices of Q, and there are 20 outlets (5 at each of the four ends of Q), where a terminal can escape from Q along a path to reach its pair. Since all terminals cannot leave Q, there is a pair π_j, $1 \le j \le 6$, that is linked by a path P_j inside Q. Let C be the set of 12 edges going out from the 3×3 square in the interior of Q. Observe that the path $P_j \subset Q$ uses (at least) two edges of C, furthermore, nine paths from the terminals in the interior of Q need to use 9 edges of C. Therefore, there is no other pair π_i, $i \ne j$, that is linked with a path inside Q.

Using the labels in Fig. 2, by symmetry, we may assume that $j = 1$ or 2. In each case P_j uses one edge among the three edges of C between $H = \{s_1, s_2, s_3\}$ and $Q - H$. Thus only four terminals in Q can access the five outlets at H (two terminals in H and two not in H), and the same is true for the five outlets at $\{t_1, t_2, t_3\}$. Thus there remain $2 \times 4 + 2 \times 5 = 18$ outlets available to escape for the $5 \times 2 + 9 = 19$ terminals in Q, a contradiction. □

For $1 \le i \le k$, let $\pi_i = \{s_i, t_i\}$ be an arbitrary *pairing* of $2k$ terminals in the grid G. Let T be the set of those $2k$ terminals and include T into a (smallest) rectangle $R \subset G$. A linkage for the k pairs can be obtained by a general two-stage procedure. This procedure is used in the proof of Proposition 2.

In the first stage we take the linkage of some pairs inside R, if we need; then every unlinked terminal $u \in T$ is mapped to a vertical or horizontal halfline $\ell(u)$ going out from R with endpoint at u, and such that all those halflines are pairwise edge-disjoint and disjoint from the paths used in the initial linkage in R. To obtain such a mapping we shall use Hall's classical matching theorem (see [5]).

In the second stage we take k pairwise vertex-disjoint cycles $F_i \subset G$, $1 \le i \le k$, around R. Each of these cycles intersect all the halflines assigned to unlinked terminals in the first stage and going out from R. For an unlinked terminal $u \in \pi_i$, $1 \le i \le k$, let $u^* \in \ell(u) \cap F_i$. Then the linkage for any unlinked pair π_i is completed along F_i between s_i^* and t_i^*, referred as to the 'mates' of s_i and t_i. A linkage for $k = 4$ obtained by this procedure is shown in Fig. 3.