Information Processing Letters 136 (2018) 37–40

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

The connectivity of generalized graph products

Xiaomin Li [a,*,1], Lan Lei [a,1], Hong-Jian Lai [b]

[a] Department of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, PR China
[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA

ARTICLE INFO

Article history:
Received 3 December 2016
Accepted 21 March 2018
Available online 28 March 2018
Communicated by Jinhui Xu

Keywords:
Interconnection networks
Connectivity
Generalized prisms
Generalized product

ABSTRACT

Bermond et al. (1984) [2] introduced a generalized product of graphs to model and construct large reliable networks under optimal conditions. This model includes the generalized prisms (also known as the permutation graphs). Piazza and Ringeisen (1991) [13] studied the optimal connectivity of generalized prisms and Lai (1995) [8] investigated the maximum subgraph connectivity of the generalized prisms. Li et al. extended these results to generalized products of trees. In this paper, we investigate the maximum subgraph connectivity of generalized product of graphs, which extends the previous results mentioned above, and obtain sufficient conditions to warrant the construction of large survivable networks via generalized products. Sharpness of our results are addressed.

1. Introduction

We follow Bondy and Murty [1] for undefined notation and terminology, and consider only finite loopless graphs in this note. For a graph G, $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$ denote the connectivity, the edge-connectivity and the minimum degree of G, respectively. For an integer $n > 0$, define $\overline{n} = \{1, 2, ..., n\}$, and following [7], let $A(\overline{n})$ denote the group of permutations on $\overline{n}$. When $\overline{n}$ is understood from the context, we often use S_n for $A(\overline{n})$. Let G be a graph with $V(G) = \{x_1, x_2, ..., x_n\}$ and G_1 and G_2 be two copies of G with $V(G_j) = \{x_1^j, x_2^j, ..., x_n^j\}$, $1 \le j \le 2$. If $\alpha \in S_n$, then the α-**generalized prism** over G, denoted by $\alpha(G)$, is the graph obtained from the disjoint union of G_1 and G_2 together with the additional edges $\{x_i^1 x_{\alpha(i)}^2 \mid 1 \le i \le n\}$. Prior results on generalized prisms can be found in [4,5,13], among others.

Let $U(G) = \min\{|S| + |V(C)|\}$, where the minimum is taken over all vertex-cuts S of G and all nonempty components C of $G - S$. The following have been proved.

Theorem 1.1. *Let G be a connected graph of order $n > 1$. Each of the following holds for any permutation $\alpha \in S_n$.*
(i) (Piazza and Ringeisen [12,13]) $\min\{2\kappa(G), U(G)\} \le \kappa(\alpha(G)) \le U(G)$.
(ii) ([8]) $\min\{2\kappa(G), \delta(G) + 1\} \le \kappa(\alpha(G)) \le \delta(G) + 1$.
(iii) (Formula (2) in [8]) $U(G) = \delta(G) + 1$.

Theorem 1.2. *Let G be a connected graph of order $n > 1$. Each of the following holds.*
(i) (Piazza and Ringeisen [12,13]) If $\kappa(G) = \delta(G)$, then $\kappa(\alpha(G)) = \delta(\alpha(G))$ for any $\alpha \in S_n$.
(ii) (Piazza and Ringeisen [12,13]) If $\kappa'(G) = \delta(G)$, then $\kappa'(\alpha(G)) = \delta(\alpha(G))$ for any $\alpha \in S_n$.
(iii) (Corollary 2.2 in [8]) $\kappa(\alpha(G)) = \delta(\alpha(G))$, if and only if $2\kappa(G) \ge \delta(G) + 1$ for any $\alpha \in S_n$.

* Corresponding author.
E-mail address: lxm@ctbu.edu.cn (X. Li).

[1] This research was partially supported by the Basic and Advanced Research Project of CQ CSTC [cstc2014jcyjA00033], Science Foundation of Chongqing Education Committee (KJ1400630) and National Natural Science Foundation of China Project (11471059).

https://doi.org/10.1016/j.ipl.2018.03.014
0020-0190/

(iv) (Corollary 2.2 in [8]) $\kappa'(\alpha(G)) = \delta(\alpha(G))$, if and only if $2\kappa'(G) \geq \delta(G) + 1$ for any $\alpha \in S_n$.

Let $\varphi(G)$ denote a graphical function and define $\overline{\varphi}(G)$ to be the maximum value of $\varphi(H)$ taken over all subgraphs H of G. As indicated in [6], for certain network reliability measures φ, networks G with $\varphi(G) = \overline{\varphi}(G)$ are important for network survivability, and so the study of $\overline{\varphi}(G)$ is of interest. Mader [10] and Matula [11] first studied $\overline{\kappa}(G)$ and $\overline{\kappa}'(G)$ for a graph G. A permutation graph version is proved in [8].

Theorem 1.3. *(Corollary 2.3 in [8]) Let G be a connected graph with n vertices. Then each of the following holds.*
(i) If $\kappa(G) = \overline{\delta}(G)$, then $\kappa(\alpha(G)) = \overline{\delta}(\alpha(G))$ for any $\alpha \in S_n$.
(ii) If $\kappa'(G) = \overline{\delta}(G)$, then $\kappa'(\alpha(G)) = \overline{\delta}(\alpha(G))$ for any $\alpha \in S_n$.

Bermond et al. in [2] introduced the generalized product of graphs, which extends the notion of generalized prisms.

Definition 1.4. *([2]) Let G and L be connected graphs with $n = |V(G)|$, $D = D(L)$ be an orientation of L, and $f : A(D) \to S_n$ be a mapping from the arc set $A(D)$ to the permutation group. We define the generalized product of G and L, as follows.*
(1) Denote $V(D) = \{u_1, u_2, \cdots u_m\}$, and $A(D) = \{e_1, e_2, \cdots, e_\ell\}$. Following [1], an arc $e \in A(D)$ oriented from a vertex u_s to a vertex u_t is denoted by (u_s, u_t).
(2) Denote $V(G) = \{v_1, v_2, \cdots, v_n\}$, and let $G_1, G_2, \cdots, G_m$ be vertex-disjoint copies of G such that for each j with $1 \leq j \leq m$, $V(G_j) = \{v_1^j, v_2^j, \cdots, v_n^j\}$ and the mapping $v_i \mapsto v_i^j$ is a graph isomorphism between G and G_j. For each $u_i \in V(D)$, we use G_i to denote the copy of G corresponding to the vertex u_i.
(3) For the mapping $f : A(D) \to S_n$, if $e = (u_i, u_j) \in A(D)$, and if $\alpha = f(e) \in S_n$, then define

$$E_{ij} = E_{ij}^f = \{v_t^i v_{\alpha(t)}^j : 1 \leq t \leq n\}.$$

For notational convenience in the proofs of the main results, we also use $f(v_t^i)$ to denote $v_{\alpha(t)}^j$.
(4) The generalized permutation graph $G^{D,f}$ is the graph with vertex set $V(G^{D,f}) = \bigcup_{j=1}^m V(G_j)$ and edge set

$$E(G^{D,f}) = (\bigcup_{j=1}^m E(G_j)) \bigcup (\bigcup_{(u_i,u_j)\in A(D)} E_{ij}).$$

When $L = K_2$, $A(D) = \{(u_1, u_2)\}$ and $f(A(D)) = \{\alpha\}$, we have $G^{D,f} = \alpha(G)$ (called a permutation graph in [5]) and as defined in [13]. The following observation follows immediately from Definition 1.4.

Observation 1.5. *Let D' be an orientation of L obtained from D by reversing an oriented edge $e = (u_i, u_j)$, and let $\alpha = f(e)$. Define $f' : A(D) \mapsto S_n$ to be a map that agrees with f on $A(D) - \{e\}$ and $f'(e) = \alpha^{-1}$. Then $G^{D,f} = G^{D',f'}$.*

By Observation 1.5, when the orientation D is understood from the context or not emphasized, we shall use $G^{L,f}$ without specifically indicating the orientation D.

Lemma 1.6. *(Bermond, Delorme and Farhi [2]) Let G be a connected graph and for any orientation D of L, $f : A(D) \mapsto S_n$ be a map. Then $\delta(G^{L,f}) = \delta(G) + \delta(L)$.*

The relationship between the connectivity and edge connectivity of $G^{D,f}$ and graph invariants of G have been studied in [3,9].

Theorem 1.7. *(Balbuena, Garcia-Vazquez and X. Marcote [3]) If G and L are two connected graphs, then for any orientation D of L and any $f : A(D) \mapsto S_n$,*

$$\min\{|V(G)|\kappa(L), (\delta(L)+1)\kappa(G), \delta(G)+\delta(L)\} \\ \leq \kappa(G^{L,f}) \leq \delta(G) + \delta(L).$$

Theorem 1.8. *(Li, Li and Li [9]) If G is a connected graph with $n = |V(G)|$ and L is a tree with $m = |V(L)|$, then for any orientation D of L and any $f : A(D) \mapsto S_n$,*

$$\min\{m\kappa'(G), \delta(G)+1\} \leq \kappa'(G^{L,f}) \leq \delta(G) + 1.$$

The current research is motivated by the theorems listed above. The purpose of this note is to extend Theorems 1.2 and 1.3 to the generalized products of graphs, as recalled in Definition 1.4. Our main results are presented below.

Theorem 1.9. *Let G and L be two connected graphs with $|V(G)| = n$ and $|V(L)| = m$, and for any orientation D of L, $f : A(D) \mapsto S_n$ be an arbitrary mapping. Each of the following holds.*
(i) If $\kappa(G) = \delta(G)$ and $\kappa(L) = \delta(L)$, then $\kappa(G^{L,f}) = \delta(G^{L,f})$.
(ii) If $\kappa'(G) = \delta(G)$ and $\kappa'(L) = \delta(L)$, then $\kappa'(G^{L,f}) = \delta(G^{L,f})$.
(iii) Suppose that $\kappa'(L) = \delta(L)$. Then for any $f : A(D) \mapsto S_n$ and any orientation D of L, $\kappa'(G^{L,f}) = \delta(G^{L,f})$ if and only if $m\kappa'(G) \geq \delta(G) + \kappa'(L)$.

Theorem 1.10. *Let G and L be two connected graphs with $|V(G)| = n$ and $|V(L)| = m$, and for any orientation D of L, $f : A(D) \mapsto S_n$ be an arbitrary mapping. Each of the following holds.*
(i) If $\kappa(G) = \overline{\delta}(G)$ and $\kappa(L) = \overline{\delta}(L)$, then $\kappa(G^{L,f}) = \overline{\delta}(G^{L,f})$.
(ii) If $\kappa'(G) = \overline{\delta}(G)$ and $\kappa'(L) = \overline{\delta}(L)$, then $\kappa'(G^{L,f}) = \overline{\delta}(G^{L,f})$.

Note that Theorem 1.9 and Theorem 1.10 present a model on constructing large network: for a graph L with property P, if a graph G satisfies property P, then $G^{L,f}$ satisfies property P also. That is, $G^{L,f}$ inherits the property P.

The proofs of the main theorems are presented in the next section.

2. Proof of the main results

Let G and L be two connected graphs with $n = |V(G)|$ and $m = |V(L)|$. Throughout this section, we shall use the notation in Definition 1.4. In particular, $G_1, G_2, ..., G_m$ are vertex-disjoint copies of G such that for each j with