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The optimal routing of augmented cubes ☆

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ABSTRACT

A routing in a graph G is a set of paths connecting each ordered pair of vertices. Load of an edge e is the number of times it appears on these paths. The edge-forwarding index of G is the smallest of maximum loads over all routings. Augmented cube of dimension n, AQ_n, is the Cayley graph $(\mathbb{Z}_2^n, \{e_1, e_2, \dots, e_n, J_2, \dots, J_n\})$ where e_i's are the vectors of the standard basis and $J_i = \sum_{j=n-i+1}^{n} e_j$. S.A. Choudum and V. Sunitha showed that the greedy algorithm provides a shortest path between each pair of vertices of AQ_n. Min Xu and Jun-Ming Xu claimed that this routing also proves that the edge-forwarding index of AQ_n is 2^{n-1}. Here we disprove this claim, by showing that in this specific routing some edges are repeated nearly $\frac{4}{3}2^{n-1}$ times (to be precise, $\lfloor \frac{2^{n+1}}{3} \rfloor$ for even values of n and $\lceil \frac{2^{n+1}}{3} \rceil$ for odd values of n). However, by providing other routings, we prove that 2^{n-1} is indeed the edge-forwarding index of AQ_n.

1. Introduction

Heydemann et al. [4] introduced the notation of the edge-forwarding index. Given a connected graph G of order n, a *routing* R is a set of $n(n-1)$ elementary paths $R(u, v)$ specified for every ordered pair (u, v) of vertices of G. The *load* $\pi(G, R, e)$ of an edge e with respect to R is defined as the number of paths of R going through e. The edge-forwarding index of G with respect to R, denoted $\pi(G, R)$, is the maximum load of all edges of G. The minimum edge-forwarding index over all possible routings is denoted by $\pi(G)$ and is called the *edge-forwarding index* of G. A routing for which $\pi(G)$ is attained is called *optimal*. If each path in R is a shortest path connecting its two ends, then the routing R is said to be a *minimal routing*.

In [1], Choudum and Sunitha introduced the augmented cubes and studied them for application in routing and broadcasting procedures. They provided several equivalent definitions of these graphs. Here we present two definitions, an inductive definition and a definition as a Cayley graph. Let $\mathbb{Z}_2^n$ be the n-dimensional binary group and let $x \oplus_2 y$ denote the binary sum of vectors x and y in $\mathbb{Z}_2^n$. The inductive definition of the augmented cubes is as follows.

Definition 1. The augmented cubes of dimensions 1 and 2 are simply the complete graphs on two and four vertices, respectively. For $n \geq 2$ the augmented cube of dimension n, denoted AQ_n, is a graph on $\mathbb{Z}_2^n$ built from two copies AQ_{n-1}^0 and AQ_{n-1}^1 of AQ_{n-1} as follows: vertices of AQ_{n-1}^0 (respectively AQ_{n-1}^1) are viewed as elements of $\mathbb{Z}_2^n$ by adding a 0 (a 1) as the first coordinate. A ver-

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tex $0x$ in AQ^0_{n-1} is adjacent, furthermore, to vertices $1x$ and $(0x \oplus_2 J)$ in AQ^1_{n-1} where J is the all-1 vector in $\mathbb{Z}_2^n$.

Let Γ be an additive group, and let S be a symmetric subset of Γ (i.e., $-S = S$) such that 0 is not in S. The *Cayley graph* (Γ, S) is the graph whose vertices are elements of Γ where two vertices x and y are adjacent if $x - y \in S$.

When the binary group $\mathbb{Z}_2^n$ is considered, then for any subset X we have $X = -X$. Let $e_1^n, e_2^n, \ldots, e_n^n$ be the vectors of the standard basis of $\mathbb{Z}_2^n$, thus e_i^n is the binary vector of length n whose ith-coordinate is 1 and all other coordinates are 0. For $i \geq 2$, let J_i^n be the binary vector of length n where the last i coordinates are 1 and the first $n-i$ coordinates are 0, i.e., $J_i^n = \sum_{j=n-i+1}^{n} e_j^n$. Then the augmented cube of dimension n defined above is known to be isomorphic to the following Cayley graph.

Proposition 2. *[1] For every $n \geq 1$, AQ_n is isomorphic to the Cayley graph $(\mathbb{Z}_2^n, S_n)$, where $S_n = \{e_1^n, e_2^n, \ldots, e_n^n, J_2^n, \ldots, J_n^n\}$.*

When it is clear from the context, we write $e_1, e_2, \ldots, e_n$ and $J_2, \ldots, J_n$ in place of $e_1^n, e_2^n, \ldots, e_n^n$ and $J_2^n, \ldots, J_n^n$.

Based on Cayley graph presentation of AQ_n a minimal routing R_n of AQ_n, originally proposed in [1], is as follows: Given vertices X and Y to find a shortest path from X to Y we find the first coordinate (i) at which X and Y differ. If X and Y also differ in the following coordinate $(i+1)$, then we define $X_1 = X + J_{n-(i-1)}$ (i.e., we change the values at coordinates i and after), otherwise we define $X_1 = X + e_i$ (i.e., we change the values only at coordinate i). Continuing this process on the newly obtained vertex, we find a shortest path to Y.

In [5], the authors studied the edge-forwarding index of the augmented cubes. Among other results, they claimed that $\pi(AQ_n) = 2^{n-1}$ and that this value is obtained by the minimal routing R_n. Here we show that the latter claim is not correct, we show that the edge-forwarding index of AQ_n with respect to R_n is nearly $\frac{4}{3}2^{n-1}$ (to be precise, $\lfloor \frac{2^{n+1}}{3} \rfloor$ for even values of n and $\lceil \frac{2^{n+1}}{3} \rceil$ for odd values of n). We then introduce a new optimal routing that prove the claim $\pi(AQ_n) = 2^{n-1}$.

To present our work, we first present in Section 2 the notion of an HMS-routing which was defined by Gauyacq [3]. We show that the minimal routing R_n [1] is an HMS-routing. But its edge-forwarding index is $\lfloor \frac{2^{n+1}}{3} \rfloor$ for even values of n and $\lceil \frac{2^{n+1}}{3} \rceil$ for odd values of n. For $n = 3$, we give a routing of AQ_3 whose edge-forwarding index is 4. However we prove that any HMS-routing of AQ_3 has an edge of load 6, i.e., the edge-forwarding index of any HMS-routing of AQ_3 is 6. For $n \geq 4$, we give an HMS-routing of AQ_n whose edge-forwarding index is 2^{n-1}.

2. HMS-routings in Cayley graphs

We would like to recall the HMS-routing defined in [3]. Let Γ be an additive group which is commutative. Let S be a symmetric subset of Γ and (Γ, S) be the corresponding Cayley graph. Let 0 denote the identity element of Γ. For each γ in Γ, the permutation ϕ_γ of Γ defined by $\phi_\gamma(h) = \gamma + h$ is an automorphism of (Γ, S) (i.e. a bijection that preserves adjacency). Furthermore, observe that if P is a path connecting vertices x and y, then the image of P under ϕ_γ is a path connecting $\phi_\gamma(x)$ and $\phi_\gamma(y)$.

Given a Cayley graph $G = (\Gamma, S)$, an HMS-routing R is a routing satisfying the following. For every vertex $v \neq 0$ in Γ, the route $R(0, v)$ is any shortest path from 0 to v. For an arbitrary pair of vertices x and y in Γ the route from x to y is defined by $R(x, y) = \phi_x(R(0, y-x))$.

We denote $00\cdots0$ by 0^n. Next, we want to prove that the minimal routing R_n defined in [1] is an HMS-routing.

Observation 3. R_n is an HMS-routing of AQ_n for every $n \geq 1$.

Proof. We know from [1] that R_n provides a shortest path from 0 to x for any $x \in \mathbb{Z}_2^n$. It remains to show that $R_n(x, y) = \phi_x(R_n(0, y-x))$. But this follows from the fact that vectors a and b of $\mathbb{Z}_2^n$ differs in the same coordinates as the vectors $\phi_\gamma(a)$ and $\phi_\gamma(b)$. It would then be enough to take $a = x$, $b = y$ and $\gamma = x$ (noting that $x = -x$ in $\mathbb{Z}_2^n$). $\square$

We recall some notations from [3]. Let (Γ, S) be a Cayley graph. If $v = u + s$ with $s \in S$, then assign the *type* s to the ordered pair (u, v), the type $-s$ to the ordered pair (v, u) and say that the edge $\{u, v\}$ is of type s or of type $-s$. A path $P = (u_0, u_1, \ldots, u_k)$ in (Γ, S) is uniquely determined by its initial vertex u_0 and the sequence $(s_1, s_2, \ldots, s_k)$ of the types of pairs of adjacent vertices. In other words, for $1 \leq i \leq k$, $u_i = u_{i-1} + s_i$. Denote by $t_s(P)$ the number of times the generator s occurs in the sequence $(s_1, s_2, \ldots, s_k)$. The following theorem was observed by Gauyacq [3].

Proposition 4. *[3] Let R be an HMS-routing in a Cayley graph (Γ, S). Let 0 be the identity and e be an edge of type s in (Γ, S). The load of e for the routing R is*

$$\pi(e) = \sum_{v \in \Gamma, v \neq 0} t_s(R(0, v)) + \sum_{v \in \Gamma, v \neq 0} t_{-s}(R(0, v)).$$

In the augmented cubes AQ_n, the identity is 0^n and $s = -s$ for any $s \in S$. We get the following corollary immediately.

Corollary 5. *Let R be an HMS-routing in AQ_n. The load of an edge e of type s for the routing R is*

$$\pi(e) = 2 \sum_{X \in V(AQ_n), X \neq 0^n} t_s(R(0^n, X)).$$

The corollary shows that, for an HMS-routing, the load of an edge depends only on its type. In a Cayley graph the problem of finding a shortest path from 0 to a vertex v is equivalent to finding a minimum length generating sequence for the element v. The problem of finding a minimum length generating sequence is known to be NP-hard [2]. More precisely, in [2] it is shown that given a set S of