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Use of logical models for proving infeasibility in term rewriting ☆

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ABSTRACT

Given a (Conditional) Rewrite System $\mathcal{R}$ and terms s and t, we consider the following problem: is there a substitution σ instantiating the variables in s and t such that the *reachability test* $\sigma(s) \to^*_{\mathcal{R}} \sigma(t)$ succeeds? If such a substitution does *not* exist, we say that the problem is *infeasible*; otherwise, we call it *feasible*. Similarly, we can consider *reducibility*, involving a *single* rewriting step. In term rewriting, a number of important problems involve such *infeasibility tests* (e.g., *confluence* and *termination* analysis). We show how to recast infeasibility tests into the problem of *finding a model* of a set of (first-order) sentences representing the operational semantics of $\mathcal{R}$ together with some additional sentences representing the considered property which is formulated as an infeasibility test.

1. Introduction

Conditional Term Rewriting Systems (CTRSs) [19, Section 7] consist of rules $\ell \to r \Leftarrow c$, where the *conditional part* c is a (possibly empty) sequence $s_1 \approx t_n, \dots, s_n \approx t_n$ of *conditions* whose *satisfaction* is required before being allowed to apply a rewriting step with ℓ and r in the usual way. Several *interpretations* of the satisfiability of conditions are possible [19, Definition 7.1.3]. For instance, dealing with *oriented* CTRSs, the evaluation of c with respect to a substitution σ consists of testing the instances of s_i and t_i for *reachability*, i.e., checking whether $\sigma(s_i)$ rewrites into $\sigma(t_i)$, written $\sigma(s_i) \to^*_{\mathcal{R}} \sigma(t_i)$, for all $1 \le i \le n$.

Given $n > 0$, a sequence $(s_i \bowtie_i t_i)_{i=1}^n$ of *rewriting goals* $s_i \bowtie_i t_i$, where s_i and t_i are terms, and $\bowtie_i$ are *predicate symbols* $\to$ or $\to^*$ is called a *feasibility sequence*. Such a sequence is $\mathcal{R}$*-feasible* if there is a substitution σ such that the instantiated goals $\sigma(s_i) \bowtie_i \sigma(t_i)$ are satisfied when $\bowtie_i$ is interpreted as the one-step or rewriting relations $\to_{\mathcal{R}}$ and $\to^*_{\mathcal{R}}$ for $\mathcal{R}$, respectively; otherwise, the sequence is called $\mathcal{R}$*-infeasible* (see Section 2 for a formal definition).

Example 1. Consider the CTRS $\mathcal{R}$ [19, Example 7.2.45]:

$$\mathsf{a} \to \mathsf{a} \Leftarrow \mathsf{b} \approx x, \mathsf{c} \approx x \tag{1}$$

$$\mathsf{b} \to \mathsf{d} \Leftarrow \mathsf{d} \approx x, \mathsf{e} \approx x \tag{2}$$

$$\mathsf{c} \to \mathsf{d} \Leftarrow \mathsf{d} \approx x, \mathsf{e} \approx x \tag{3}$$

where $\mathsf{a}, \dots, \mathsf{e}$ are *constants* and x is a *variable*. Since d and e are irreducible, the only way for $\mathsf{d} \to^* x, \mathsf{e} \to^* x$ to be $\mathcal{R}$-feasible is instantiating x to both d and e at the same time, which is *not* possible. Thus, (2) and (3) *cannot* be used in any rewriting step. They are called *infeasible* rules and may be removed (without changing the induced rewrite relation). Actually, (1) is infeasible too.

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$$(R)\quad \frac{}{x \to^* x} \qquad (C)\quad \frac{x_i \to y_i}{f(x_1,\ldots,x_i,\ldots,x_k) \to f(x_1,\ldots,y_i,\ldots,x_k)}\ \text{for all } f \in \mathcal{F} \text{ and } 1 \le i \le k = arity(f)$$

$$(T)\quad \frac{x \to y \quad y \to^* z}{x \to^* z} \qquad (Rl)\quad \frac{s_1 \to^* t_1 \quad \cdots \quad s_n \to^* t_n}{\ell \to r}\ \text{for } \ell \to r \Leftarrow s_1 \approx t_1,\ldots,s_n \approx t_n \in \mathcal{R}$$

Fig. 1. Inference rules for conditional rewriting with a CTRS $\mathcal{R}$ with signature $\mathcal{F}$.

$\mathcal{R}$*-infeasibility* can be used to (i) disable the use of *conditional rules* in reductions or even *remove* them,[1] (ii) discard *conditional dependency pairs* $u \to v \Leftarrow c$ in the analysis of *operational termination* of CTRSs [12], (iii) discard *conditional critical pairs* $u \downarrow v \Leftarrow c$ that arise in the analysis of *confluence* of CTRSs [19–21], (iv) prove *root-stability* of a term t (i.e., the absence of any rewriting sequence from t to an instance of a left-hand side ℓ of a rewrite rule $\ell \to r \Leftarrow c$) as the $\mathcal{R}$-infeasibility of $t \to^* \ell$ for each $\ell \to r \Leftarrow c \in \mathcal{R}$, (v) prove *irreducibility* of ground terms t (which is undecidable for CTRSs) as the $\mathcal{R}$-infeasibility of $t \to x$ for a variable x, (vi) prove the *non-joinability* of terms s and t as the $\mathcal{R}$-infeasibility of $s \to^* x, t \to^* x$ (with x not occurring in s or t), or (vii) discard *arcs* in the *dependency* graphs that are obtained during the analysis of termination using dependency pairs, see, e.g., [2] for TRSs and [11] for CTRSs.

In Section 3, we prove that $\mathcal{R}$-infeasibility problems can be translated into the problem of finding a *model* $\mathcal{A}$ of the set of sentences $\overline{\mathcal{R}}$ representing the operational semantics of the CTRS $\mathcal{R}$ plus a sentence $\neg(\exists\vec{x})\ \bigwedge_{i=1}^n s_i \bowtie_i t_i$ where all symbols (including $\to$ and $\to^*$ as predicate symbols) can be freely interpreted in a first-order structure $\mathcal{A}$. In Section 4 we show by means of examples how to apply our method to problems (i)–(vii). We assume familiarity with the basic notions, terminology and notations of (conditional) term rewriting (see, e.g., [3,19] for TRSs and [19, Section 7] for CTRSs) and first-order logic [16].

The research in this paper was first presented in [8] (a restricted, non-systematic use in proofs of operational termination of CTRSs is sketched in [7, Section 11.1]) and then settled by the first author in a first-order logic framework in [6]. In this paper we have extended the treatment of [8] to more general properties of rewrite systems (e.g., reducibility or root-stability, see Section 4.3). Also, in contrast to [6], we show that focusing on CTRSs and term rewriting enables the use of specific refinements available for CTRSs only (e.g., usable rules, see Section 2). This allows us to deal with more applications and examples. Interestingly, our semantic approach together with the aforementioned improvements applies to *all* the examples solved in papers developing different specific techniques to deal with problems (i)–(vii) [1,17,20,21].

[1] Sometimes, infeasible rules *cannot* be removed without changing relevant properties of a CTRS. For instance, $\mathcal{R} = \{\mathsf{b} \to \mathsf{c}, \mathsf{a} \to \mathsf{b} \Leftarrow \mathsf{a} \approx \mathsf{b}\}$ is *not* operationally terminating (see Section 4.1 below) due to the conditional rule, which is *infeasible*. However, after removing it, an *operationally terminating* CTRS is obtained.

2. Infeasibility problems

Borrowing [19, Definition 7.1.8(3)] we introduce the following.

Definition 2. Let $\mathcal{R}$ be a CTRS. A sequence $(s_i \bowtie_i t_i)_{i=1}^n$, where s_i and t_i are terms and $\bowtie_i \in \{\to, \to^*\}$ for all $1 \le i \le n$ is called a *feasibility sequence*. It is $\mathcal{R}$*-feasible* if there is a substitution σ such that, for all $1 \le i \le n$, $\sigma(s_i) \to_{\mathcal{R}} \sigma(t_i)$ if $\bowtie_i$ is $\to$, and $\sigma(s_i) \to^*_{\mathcal{R}} \sigma(t_i)$ if $\bowtie_i$ is $\to^*$. Otherwise, it is $\mathcal{R}$*-infeasible*.

In Definition 2 we write $s \to^*_{\mathcal{R}} t$ or $s \to_{\mathcal{R}} t$ for terms s and t iff there is a proof tree for $s \to^* t$ (resp. $s \to t$) using $\mathcal{R}$ in the inference system of Fig. 1 [9]. All rules in the inference system in Fig. 1 are *schematic* in that each inference rule $\frac{B_1 \cdots B_n}{A}$ can be used under any *instance* $\frac{\sigma(B_1) \cdots \sigma(B_n)}{\sigma(A)}$ of the rule by a substitution σ. For instance, (Rl) actually establishes that, for every rule $\ell \to r \Leftarrow s_1 \approx t_1,\ldots,s_n \approx t_n$ in the CTRS $\mathcal{R}$, every instance $\sigma(\ell)$ by a substitution σ rewrites into $\sigma(r)$ provided that, for each $s_i \approx t_i$, with $1 \le i \le n$, the reachability condition $\sigma(s_i) \to^* \sigma(t_i)$ can be *proved*. We have the following:

Proposition 3. *Let $\mathcal{R}$ be a CTRS. A feasibility sequence $(s_i \bowtie_i t_i)_{i=1}^n$ is $\mathcal{R}$-infeasible if $(s_i \bowtie_i t_i)_{i \in I}$ is $\mathcal{R}$-infeasible for some $I \subseteq \{1,\ldots,n\}$.*

Aoto observes the following: for TRSs $\mathcal{R}$, the so-called *usable rules for reachability* associated to a term s, $\mathcal{U}(\mathcal{R},s)$ [1, Definition 3], are the only ones we need in any rewriting sequence starting from s, i.e., $s \to^*_{\mathcal{R}} t$ iff $s \to^*_{\mathcal{U}(\mathcal{R},s)} t$ [1, Lemma 4]. This also holds for the usable rules $\mathcal{U}(\mathcal{R},s)$ for CTRSs $\mathcal{R}$ and terms s in [13, Definition 11] that we introduce below. First let

$$RULES(\mathcal{R},t) = \{\ell \to r \Leftarrow c \in R \mid \exists p \in \mathcal{P}os(t),\ root(\ell) = root(t|_p)\}$$

Note that $RULES(\mathcal{R},t)$ contains the rules of $\mathcal{R}$ which are potentially applicable to the subterms in t. Now, the set of *usable rules* for a term t is:

$$\mathcal{U}(\mathcal{R},t) = RULES(\mathcal{R},t) \cup \bigcup_{l \to r \Leftarrow c \in RULES(\mathcal{R},t)} \left(\mathcal{U}(\mathcal{R}^\sharp, r) \cup \bigcup_{s \approx t \in c} \mathcal{U}(\mathcal{R}^\sharp, s) \right)$$