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The direct sum of universal relations ☆

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ABSTRACT

The universal relation is the communication problem in which Alice and Bob get as inputs two distinct strings, and they are required to find a coordinate on which the strings differ. The study of this problem is motivated by its connection to Karchmer–Wigderson relations [12], which are communication problems that are tightly related to circuit-depth lower bounds.

In this paper, we prove a direct sum theorem for the universal relation, namely, we prove that solving m independent instances of the universal relation is m times harder than solving a single instance. More specifically, it is known that the deterministic communication complexity of the universal relation is at least n. We prove that the deterministic communication complexity of solving m independent instances of the universal relation is at least $m \cdot (n - O(\log m))$.

1. Introduction

The direct-sum question is a classical question that asks whether performing a task on m independent inputs is m times harder than performing it on a single input. This natural question was studied in a variety of computational models (see, e.g., [20,14,6,18,2,3]), and the answer turns out to be positive in some models and negative in others. Karchmer, Raz, and Wigderson [11] initiated the study of this question in the setting of communication complexity. One motivation was a connection that they observed between the direct-sum question for the *deterministic* communication complexity of *relations* and the circuit-depth complexity of functions.

Later works have made considerable progress in the study of direct sum for *randomized* communication complexity [1,7] and for the deterministic communication complexity of *functions* [5]. However, there is only one[2] known result on the direct-sum question in the original setting of [11] — *deterministic* protocols for *relations*: a direct-sum theorem for a relation that is connected to the set covering problem, which appears in the original paper of [11]. In this work, we provide another example for such a direct-sum theorem, namely, for the universal relation.

The universal relation is the following communication problem: Alice and Bob get two *distinct* strings

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[2] There are also a few examples of relations for which direct-sum theorems on the deterministic complexity follow trivially from the corresponding results on randomized complexity: This happens when the deterministic complexity of the relation is equal to its randomized complexity, and there is a direct-sum theorem for the randomized complexity. This is the case, for example, for the monotone Karchmer–Wigderson relations of the clique and matching functions [17].

However, we are interested in direct-sum theorems on deterministic complexity that are "non-trivial" in the sense that they do not follow directly from results on randomized complexity. This is the case for the universal relation, whose randomized complexity is much smaller than its deterministic complexity ($O(\log n)$ vs. n [16]).

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$x, y \in \{0,1\}^n$, and they are required to find a coordinate $j \in [n]$ such that $x_j \neq y_j$. This problem is a simplified version of Karchmer–Wigderson relations [12], which are communication problems that are tightly related to circuit-depth lower bounds. The universal relation was introduced by [11] in the hope that a better understanding of the universal relation would lead to progress in the study of Karchmer–Wigderson relations, and hence to better circuit-depth lower bounds. It is known that the deterministic communication complexity of the universal relation is exactly $n+1$ [19]. We prove the following result.

Theorem 1. *For every $m \geq 4$, the deterministic communication complexity of solving m independent instances of the universal relation over n bits is at least $m \cdot (n - 2\log(m) - 8)$.*

Our proof is based on the works of Edmonds et al. [4] and Raz and McKenzie [15] on composition theorems. In this work, we show how their techniques can be applied to the setting of direct-sum theorems. We hope that our ideas will lead to more direct-sum results in the future.

Remark 2. We note that a similar result is stated without proof at the end of [11].

Remark 3. Following an earlier version of this work [13], Alexander Kozachinsky [10] pointed out a simple rank argument that proves a much stronger lower bound of $m \cdot (n-1) - 1$ (which may be the proof that [11] had in mind). We describe this argument in Remark 6 below. We maintain our original (weaker and more complicated) proof in this version since we believe its ideas are valuable and might lead to new direct-sum theorems for other relations in the future.

The paper is organized as follows: In Section 2 we discuss the universal relation and its direct sum, as well as "totalized" versions of these problems which are important for our proof. We then prove Theorem 1 in Section 3.

Preliminaries For $n \in \mathbb{N}$, we denote $[n] \stackrel{\text{def}}{=} \{1, \ldots n\}$. We denote by $\{0,1\}^{m \times n}$ the set of $m \times n$ binary matrices. Given a set $I \subseteq [m]$, we denote by $\{0,1\}^{I \times n}$ the set of $|I| \times n$ binary matrices whose rows are labeled by the indices in I. Given a subset of rows $I' \subseteq I$ and a matrix $Z \in \mathcal{Z}$, we denote by $Z|_{I'}$ the projection of Z to the rows in I', and we say that Z is an *extension of the matrix $Z|_{I'}$ (to $\mathcal{Z}$)*. Given a set of matrices $\mathcal{Z} \subseteq \{0,1\}^{m \times n}$ and a set of rows $I \subseteq [m]$, we denote by $\mathcal{Z}|_I$ the set of projections of matrices in $\mathcal{Z}$ to rows in I. We use the standard definitions of communication complexity — see the book of Kushilevitz and Nisan [9] for more details.

2. The universal relation, its direct sum, and their totalizations

As explained in the introduction, the *universal relation (on n bits)*, denoted U_n, is the following communication problem: Alice and Bob get two *distinct* strings $x, y \in \{0,1\}^n$, and they are required to find a coordinate $j \in [n]$ such that $x_j \neq y_j$. It is not hard to prove that the deterministic communication complexity of this problem is at least n. On the other hand, it is interesting to note that its randomized communication complexity is at most $O(\log n)$ [16].

The direct sum of the universal relation consists of solving m independent instances of the problem. In order to streamline the presentation, it is convenient to represent the inputs to the direct sum by matrices. This leads to the following definition of the direct sum.

Definition 4. Let $m, n \in \mathbb{N}$. The *m-fold direct sum* of the universal relation on n bits, denoted $U_n^{\otimes m}$ is the communication problem in which Alice and Bob get matrices $X, Y \in \{0,1\}^{m \times n}$ that differ on every row. They are required to output a tuple $(j_1, \ldots, j_m) \in [n]^m$, such that for every row $i \in [m]$ it holds that $X_{i,j_i} \neq Y_{i,j_i}$.

Håstad and Wigderson [8] observed that it is useful to consider a variant of the universal relation, which is a total relation rather than a promise problem: In the *totalized universal relation*, denoted U'_n, Alice and Bob may be given identical strings as inputs, and in this case they should output a special "reject" symbol $\perp$. The totalized universal relation is often easier to work with than the non-totalized one. In particular, it is trivial to prove a lower bound of n on its deterministic communication complexity by a reduction from the equality function. It is not hard to see that this modification does not decrease the complexity of the universal relation by more than two bits.

Similarly, it is useful to consider a "totalization" of the direct sum of the universal relation: Alice and Bob get two arbitrary matrices $X, Y \in \{0,1\}^{m \times n}$. The parties should reject if X and Y agree on any single row, and otherwise they should output a tuple $(j_1, \ldots, j_m)$ as before.

Definition 5. Let $m, n \in \mathbb{N}$. The *totalized m-fold direct sum* of the universal relation on n bits, denoted $U_n^{\otimes m'}$, is the communication problem in which Alice and Bob get as inputs matrices $X, Y \in \{0,1\}^{m \times n}$ and behave as follows:

- If X and Y differ on every row, then Alice and Bob behave as in the (non-totalized) direct sum $U_n^{\otimes m}$.
- Otherwise, Alice and Bob output the "reject" symbol $\perp$.

Again, it is not hard to see that this modification does not decrease the complexity of the direct sum by more than $2m$ bits. Hence, to prove Theorem 1, it suffices to prove that the communication complexity of the relation $U_n^{\otimes m'}$ is at least $m \cdot (n - 2\log m - 6)$.

Remark 6. Alexander Kozachinsky [10] pointed out a simple argument that proves a lower bound of $m \cdot n$ for the totalized direct sum $U_n^{\otimes m'}$ (which in turn implies a lower bound of $m \cdot (n-1) - 1$ for the non-totalized direct sum). The argument is as follows. First, observe that it suffices to prove a lower bound of $m \cdot n$ for the following problem Alice and Bob get matrices $X, Y \in \{0,1\}^{m \times n}$ respectively, and they would like to determine whether they agree on at