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A risk–reward model with compound interest rate for non-additive two-option ski rental

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ABSTRACT

We consider the non-additive two-option ski rental problem (NTSR), which includes two options such that each Option i (for $i = 1, 2$) is characterized by a one-time cost b_i and a corresponding rental price a_i. Without loss of generality, we assume that $a_1 > a_2 \geq 0$ and $b_2 > b_1 \geq 0$. Besides, we have to pay a transition cost c if we switch from Option 1 to Option 2, where $c \geq b_2 - b_1$. We introduce the compound interest rate into the continuous version of NTSR and obtain the optimal deterministic on-line strategy by competitive analysis. Moreover, considering the risk tolerance of decision makers, we present a risk–reward strategy. In addition, we use numerical analysis to analyze the influence of risk tolerance and compound interest rate on the restricted ratio and switching time of the optimal risk–reward strategy. The results demonstrate that the competitive performance is improved when the risk tolerance and compound interest rate are considered.

1. Introduction

The leasing industry develops rapidly and demonstrates its resilience as a sunrise industry since the global economic crisis [1]. To decide whether leasing is better than buying or not, we should determine the duration the equipment will be used. However, it is usually hard to know the exact duration. Fortunately, researchers put forward the on-line algorithms and competitive analysis [2] to analyze the strategies of on-line problems. The performance of a strategy is measured by competitive ratio.

The classic instance of on-line leasing problem is the "ski rental" problem [3]. In this problem if the rental cost is 1 per day and the purchasing cost is $s(s > 1)$, then we can obtain an optimal competitive ratio $2 - 1/s$ through competitive analysis. But competitive analysis is a strong worst-case analysis and it is suitable for risk-averse investors. However, in the financial market the investors sometimes would like to undertake the risk moderately to gain more profit. So al-Binali [4] extended competitive analysis to provide a framework in which investors can develop optimal on-line strategies on the basis of their risk tolerance and forecast.

In this paper, we apply al-Binali's risk–reward model to a generalized ski rental problem, which was first formalized by Levi and Patt-Shamir [5]. They called this problem the non-additive two-option ski rental, which is NTSR for short. This problem is a non-trivial variant of the non-additive on-line problem and includes two options for leasing a facility. Each option i (for $i = 1, 2$) is characterized by the one-time cost b_i to begin using Option i and the corresponding rental price a_i, where $a_1 > a_2 \geq 0$ and $b_2 > b_1 \geq 0$. However, if we wish to switch from Option 1 to Option 2, then we should pay a transition cost c. And the transition cost satisfies $c \geq b_2 - b_1$, otherwise, the problem is reducible to the additive version. Levi and Patt-Shamir [5] discussed the simplified NTSR, where $a_i = 1 - b_i$ (for $i = 1, 2$) and proposed the optimal deterministic and

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randomized strategies as well as analyzed the competitive ratio.

As the interest rate is of great importance in the financial problem [6,7], we introduce a compound interest rate r into the risk–reward model for the NTSR. The net present value of one unit of money in t units of time is then e^{-rt} [7]. In all of the following discussions, we use the present value without providing a detailed description. Additionally, we assume $a_1 - a_2 - (b_2 - b_1)r > 0$, which is explained when the cost function is established.

The remainder of this paper is organized as follows. The basic definitions and notations are presented in Section 2. We present the optimal deterministic strategy and risk–reward strategy in Section 3 and 4, respectively. And we numerically analyze the performance of the risk–reward strategy in Section 5. Finally, we provide the conclusion in Section 6.

2. Basic definitions and notations

In this section, we present basic definitions regarding competitive analysis and al-Binali's risk–reward framework [4]. For a set Σ of inputs in a cost minimization on-line problem, the on-line player is not aware of the exact input σ, where $\sigma \in \Sigma$. However, the off-line player knows everything including the input and his cost is the lowest. We assume that the on-line player has a set of strategies $S = \{S(t) \mid t \geq 0\}$ from which to select, where t is the time when the player switches to another option. Let $Cost_S(\sigma; t)$ denote the cost of an on-line strategy $S(t)$ on input σ. The cost of the adversary's optimal algorithm is then $Cost_{off}(\sigma) = \min_{S(t)\in S} Cost_S(\sigma; t)$. Additionally, the competitive ratio of the on-line strategy $S(t)$ is $R(t) = \sup\limits_{\sigma\in\Sigma} \frac{Cost_S(\sigma;t)}{Cost_{off}(\sigma)}$. The optimal competitive ratio of this on-line problem is $R^* = \inf\limits_{S(t)\in S} R(t)$.

In al-Binali's risk–reward model, he assumes that the on-line player has a risk tolerance λ and $\lambda \geq 1$. The set of the on-line player's risk-tolerable strategies is then $I_\lambda = \{S(t) | R(t) \leq \lambda R^*\}$. He also supposes that the player has a forecast F to the inputs, where $F \subset \Sigma$. The restricted ratio of an on-line strategy $S(t)$ is then defined by $\overline{R}_F(t) = \sup\limits_{\sigma\in F} \frac{Cost_S(\sigma;t)}{Cost_{off}(\sigma)}$, which is the competitive ratio when the forecast is correct. The restricted ratio of the optimal on-line strategy called the optimal restricted ratio is $\overline{R}_F^* = \inf\limits_{S\in I_\lambda} \overline{R}_F(t)$, which can be considered as the optimal competitive ratio of the strategies in I_λ when the forecast is correct. Next, the reward of $S(t)$, as an improvement on the optimal on-line strategy, is defined by $f(t) = R^*/\overline{R}_F(t)$. An optimal risk-tolerant strategy is then $S(t_F^*) \in I_\lambda$ such that $f(t_F^*) = \sup\limits_{S(t)\in I_\lambda} f(t)$. The reward is obtained when the forecast is correct. If the forecast is false, then discussing the reward is absurd. However, the risk is within the player's risk tolerance.

3. Deterministic on-line strategy for NTSR

In this section, we present an optimal deterministic on-line strategy and give its competitive ratio for the NTSR.

First of all, we discuss the optimal off-line algorithm. We assume that the actual usage time is T. The off-line adversary is aware of T. So the optimal off-line cost is

$$
\begin{aligned}
Cost_{off}(T) &= \begin{cases} b_1 + a_1 \int_0^T e^{-r\tau} d\tau, & T < T^*; \\ b_2 + a_2 \int_0^T e^{-r\tau} d\tau, & T \geq T^*, \end{cases} \\
&= \begin{cases} b_1 + \frac{a_1}{r}(1 - e^{-rT}), & T < T^*; \\ b_2 + \frac{a_2}{r}(1 - e^{-rT}), & T \geq T^*, \end{cases}
\end{aligned} \tag{1}
$$

where $T^* = -\frac{1}{r} \ln\left[1 - \frac{(b_2-b_1)r}{a_1-a_2}\right]$. On the surface, T^* exists if and only if $a_1 - a_2 - (b_2 - b_1)r > 0$ is true. If $a_1 - a_2 - (b_2 - b_1)r \leq 0$, Option 1 is always superior to Option 2. In addition, the off-line adversary never switches between options. Because if he initially chooses Option 1 and switches to Option 2 at time t, then the total cost satisfies $b_1 + \frac{a_1}{r}(1 - e^{-rt}) + ce^{-rt} + \frac{a_2}{r}(e^{-rt} - e^{-rT}) \geq b_2 + \frac{a_2}{r}(1 - e^{-rT}) + \frac{1}{r}[a_1 - a_2 - (b_2 - b_1)r](1 - e^{-rt}) \geq b_2 + \frac{a_2}{r}(1 - e^{-rT})$. This indicates that initially choosing Option 2 is superior to the policy that first selects Option 1 then switches to Option 2.

We next provide an on-line algorithm and determine the optimal deterministic on-line strategy. The algorithm is denoted by S_t which indicates that the use of Option 2 is initiated at time t. Next, we analyze the competitive ratio of S_t when $0 \leq t \leq \infty$.

When $t = 0$, the algorithm chooses Option 2 immediately and never chooses Option 1. In this case, the adversary selects Option 2 and make $T < T^*$ to improve the competitive ratio of S_0. The competitive ratio is then $R_0(T) = \frac{b_2 + \frac{a_2}{r}(1-e^{-rT})}{b_1 + \frac{a_1}{r}(1-e^{-rT})}$. As $R_0'(T) = -\frac{(a_1b_2 - a_2b_1)r^2e^{-rT}}{[b_1r + a_1(1-e^{-rT})]^2} \leq 0$, $R_0(T)$ decreases with respect to T. The maximal competitive ratio is then attained at $T = 0$ and the optimal competitive ratio is $R_0^* = b_2/b_1$.

When $t = +\infty$, which indicates that the decision maker never switches to Option 2, the competitive ratio is $R_\infty(T) = \frac{b_1 + \frac{a_1}{r}(1-e^{-rT})}{b_2 + \frac{a_2}{r}(1-e^{-rT})}$. Now, $R_\infty'(T) = \frac{(a_1b_2 - a_2b_1)r^2e^{-rT}}{[b_2r + a_2(1-e^{-rT})]^2} \geq 0$, thus $R_\infty(T)$ is increasing in T. The adversary then causes T to be infinitely close to ∞. The worst-case competitive ratio in this situation is $R_\infty^* = \frac{a_1 + b_1r}{a_2 + b_2r}$.

When $0 < t < \infty$, we discuss the problem in two cases as follows. However, we first provide the cost function $Cost_S(t; T)$ of the strategy S_t:

$$
\begin{aligned}
Cost_S(t; T) &= \begin{cases} b_1 + a_1 \int_0^T e^{-r\tau} d\tau, & T < t; \\ b_1 + a_1 \int_0^t e^{-r\tau} d\tau + ce^{-rt} & \\ \quad + a_2 \int_t^T e^{-r\tau} d\tau, & T \geq t, \end{cases} \\
&= \begin{cases} b_1 + \frac{a_1}{r}(1 - e^{-rT}), & T < t; \\ b_1 + \frac{a_1}{r}(1 - e^{-rt}) + ce^{-rt} & \\ \quad + \frac{a_2}{r}(e^{-rt} - e^{-rT}), & T \geq t. \end{cases}
\end{aligned} \tag{2}
$$

Case 1: $0 < t < T^*$. The cost ratio $R_1(t; T)$ of the on-line and off-line strategy is