Information Processing Letters 135 (2018) 22–27

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

On the hardness of finding the geodetic number of a subcubic graph ☆

Letícia R. Bueno [a], Lucia D. Penso [b], Fábio Protti [c], Victor R. Ramos [c], Dieter Rautenbach [b], Uéverton S. Souza [c,*]

[a] CMCC, Universidade Federal do ABC, Santo André, Brazil
[b] Universität Ulm, Ulm, Germany
[c] Universidade Federal Fluminense, Niterói, Brazil

ARTICLE INFO

Article history:
Received 30 September 2017
Received in revised form 2 February 2018
Accepted 11 February 2018
Available online 13 February 2018
Communicated by B. Doerr

Keywords:
geodetic number
geodetic set
geodetic convexity
computational complexity
maximum degree three

ABSTRACT

A set of vertices D of a graph G is geodetic if every vertex of G lies in a shortest path between two vertices in D. The geodetic number of G is the minimum cardinality of a geodetic set of G, and deciding whether it is at most k is an NP-complete problem for several classes of graphs. While the problem is easy for graphs of maximum degree at most 2, we show that the problem is NP-complete for graphs of maximum degree three.

1. Introduction

Given a graph $G = (V(G), E(G))$ and two vertices u and v in G, the *distance* $d_G(u, v)$ is the length of a shortest path between u and v, or ∞ if no such path exists. The *interval* $I[u, v]$ between u and v is the set of vertices of G that belong to a shortest path between u and v. If G is connected then a vertex w belongs to $I[u, v]$ if and only if $d_G(u, v) = d_G(u, w) + d_G(w, v)$. For a set S of vertices, let the interval $I[S]$ of S be the union of the intervals $I[u, v]$ over all pairs of vertices u and v in S. A set of vertices S is *geodetic* if $I[S] \supseteq V(G)$.

The cardinality of a minimum geodetic set of G is the *geodetic number* of G, denoted by $g(G)$. The problem of deciding whether the geodetic number of a graph is at most k is NP-complete for general graphs [3] and it is known as the Geodetic Set problem. In fact, it is NP-complete even when restricted to chordal and bipartite chordal graphs [9]. However, for cographs and split graphs, $g(G)$ can be computed in linear time [9]; the authors have also provided an upper bound for unit interval graphs which is best possible. Brešar et al. [6] determined some exact values and upper bounds for the geodetic number of the Cartesian product of graphs in general. Cao et al. [7] presented exact values for the geodetic number of the Cartesian product $C_n \times C_m$ of two cycles.

Some variants of such a problem have been studied, such as the edge geodetic number [4], the geodetic number of oriented graphs [11], and the connected geodetic number [12]. The related problem of determining the *hull*

☆ An extended abstract with some results of this paper was presented at CTW 2015, the Cologne-Twente Workshop on Graphs and Combinatorial Optimization. Research partially supported by Brazilian agencies CNPq and FAPERJ.

* Corresponding author.
E-mail addresses: leticia.bueno@ufabc.edu.br (L.R. Bueno), lucia.penso@uni-ulm.de (L.D. Penso), fabio@ic.uff.br (F. Protti), victorrangelramos@id.uff.br (V.R. Ramos), dieter.rautenbach@uni-ulm.de (D. Rautenbach), ueverton@ic.uff.br (U.S. Souza).

https://doi.org/10.1016/j.ipl.2018.02.012
0020-0190/

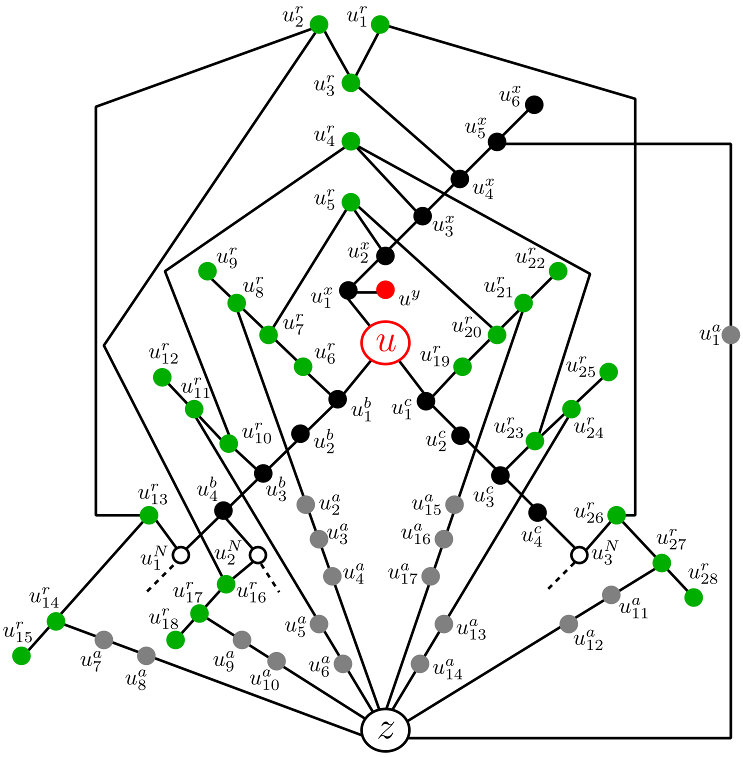

Fig. 1. Illustration for Definition 1: the graph G_u for a vertex u.

number was proved to be NP-hard for bipartite graphs [2], chordal graphs [5], and P_9-free graphs [8]. Hansberg and Volkmann [10] studied the geodetic number from the point of view of domination; they showed that the problem of deciding if a chordal or chordal bipartite graph has a geodetic domination set of size at most k is NP-complete.

In this paper we prove that the problem of deciding the geodetic number of a graph is NP-complete even for subcubic graphs, i.e., graphs with maximum degree 3.

2. The geodetic number problem for subcubic graphs

In this section, we prove that Geodetic Set is NP-complete for subcubic graphs. We organize the proof in two parts. In the former we show the NP-completeness of the problem for graphs with exactly one vertex with degree greater than three. In the latter we show how we can replace this single vertex with large degree. Thus, using a similar strategy but with more complex gadgets we prove that Geodetic Set is NP-complete for subcubic graphs.

Let the *union* of graphs G_1 and G_2 be the graph $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, and the *intersection* of G_1 and G_2 the graph $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$. In addition, let L_G be the set of pendant vertices (vertices with degree one) of G.

2.1. Graphs with only one vertex with degree greater than three

In order to show that such a problem is NP-complete for graphs with degree at most 3, we first present some definitions and preliminary results.

Definition 1. For a vertex u, let G_u arise from an empty graph by adding u, a special vertex z, and performing the following steps (see Fig. 1):

- Add paths $uu^x_1u^y$, $u^x_1u^x_2u^x_3u^x_4u^x_5u^x_6$, and $uu^c_1u^c_2u^c_3u^c_4u^N_3u^r_{26}u^r_{27}u^r_{28}$,
- add paths $uu^b_1u^b_2u^b_3u^b_4u^N_1u^r_{13}u^r_{14}u^r_{15}$, and $u^b_4u^N_2u^r_{16}u^r_{17}u^r_{18}$,
- add edges $u^r_5u^r_{20}$, $u^r_4u^r_{23}$, and $u^r_2u^r_{16}$;
- add paths $u^x_2u^r_5u^r_7$, $u^x_3u^r_4u^r_{10}$, $u^x_4u^r_3u^r_2u^r_{13}$, and $u^r_3u^r_1u^r_{26}$,
- add paths $u^b_1u^r_6u^r_7u^r_8u^r_9$, $u^c_1u^r_{19}u^r_{20}u^r_{21}u^r_{22}$, $u^b_3u^r_{10}u^r_{11}u^r_{12}$, and $u^c_3u^r_{23}u^r_{24}u^r_{25}$,
- add paths $u^x_5u^a_1z$, $u^r_8u^a_2u^a_3u^a_4z$, $u^r_{21}u^a_{15}u^a_{16}u^a_{17}z$, and $u^r_{11}u^a_5u^a_6z$,
- add paths $u^r_{24}u^a_{13}u^a_{14}z$, $u^r_{17}u^a_9u^a_{10}z$, $u^r_{27}u^a_{11}u^a_{12}z$, and $u^r_{14}u^a_7u^a_8z$.

Fig. 1 illustrates a graph G_u constructed as previously described.

Proposition 2. *Let G_u be the graph as described in Definition 1. Then it holds that $I[L_{G_u}] = V(G_u) \setminus \{u\}$.*

Proposition 3. *Let G_u be the graph as described in Definition 1. The vertices in $V(G_u) \setminus \{\{u\} \cup \{u^a_i | 1 \le i \le 17\}\}$ belong to minimum paths between pendant vertices that do not traverse z.*

One can verify Propositions 2 and 3 by analyzing all minimum paths between pendant vertices of G_u. Table 1 summarizes the main information.