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Total coloring of rooted path graphs ☆

Martin Charles Golumbic

Caesarea Rothschild Institute and the Department of Computer Science, University of Haifa, Mount Carmel, Haifa 31905, Israel

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ABSTRACT

A *total coloring* of a graph is an assignment of colors to both its vertices and edges so that adjacent or incident elements acquire distinct colors. In this note, we give a simple greedy algorithm to totally color a rooted path graph G with at most $\Delta(G)+2$ colors, where $\Delta(G)$ is the maximum vertex degree of G. Our algorithm is inspired by a method by Bojarshinov (2001) [3] for interval graphs and provides a new proof that the Total Coloring Conjecture, posed independently by Behzad (1965) [1] and Vizing (1968) [15], holds for rooted path graphs. In the process, we also prove a useful property of greedy neighborhood coloring for chordal graphs.

1. Introduction

A graph is a *rooted path graph* if it has an intersection model consisting of directed paths in a rooted directed tree [4,13]. The class of rooted path graphs contains all interval graphs and is contained in the class of strongly chordal graphs [5,9]. A *total coloring* of a graph is an assignment of colors to both its vertices and edges so that adjacent or incident elements acquire distinct colors. The least number of colors sufficient for a total coloring of a graph G is called its *total chromatic number* and is denoted by $\chi''(G)$. For example, the cycles C_4 and C_5 require four colors for a total coloring, but C_6 can be totally colored with only three colors.

It is clear that $\chi''(G) \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G, since a vertex needs one color for itself and one color each for the edges it touches. This bound is achieved exactly for the triangle K_3, the cycle C_6, trees with more than one edge, and many other graphs. However, as we saw above, a total coloring of the cycles C_4 and C_5 require $\Delta(G)+2$ colors.

The *Total Coloring Conjecture* (TCC), posed independently by Behzad [1] and Vizing [15], states that *every simple graph G has a total coloring with $\Delta(G)+2$ colors*. This has become one of the most challenging open problems in graph theory.

The Total Coloring Conjecture has been shown to hold for several subfamilies of chordal graphs, namely, interval graphs [3], split graphs [7], and strongly chordal graphs [8]. It has also been shown to hold for additional subfamilies of graphs which are not subfamilies of chordal graphs, namely, unichord-free graphs [11] and dually chordal graphs [8]. We recall that every strongly chordal graph is a dually chordal graph [5]. The TCC is not (yet) known to hold for chordal graphs.

In this note, we give a simple greedy algorithm to totally color a rooted path graph with at most $\Delta(G)+2$ colors, which gives a new proof that the Total Coloring Conjecture holds for rooted path graphs. In the case when $\Delta(G)$ is an even number, the total coloring uses $\Delta(G)+1$ colors. Our algorithm is inspired by a method by Bojarshinov [3] for interval graphs, which relies on the characteristic maximal cliques sequence. Our algorithm differs from the method presented in [8], which considers the square

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E-mail address: golumbic@cs.haifa.ac.il.

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graph G^2 of G. In the process, we prove a useful property of chordal graphs, namely, that greedy neighborhood coloring using the reverse of a perfect elimination ordering produces a proper coloring of the vertices.

2. Preliminaries

The case of totally coloring complete graphs is of particular importance for our method.

Theorem 1. *Let K_k be the complete graph on k vertices. Then*

$$\chi''(K_k) = \begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}$$

A lovely construction[1] proving Theorem 1—by rotating copies of a maximal matching and single vertex in the odd case, and adding a dummy vertex to turn the even case into an odd case—allows one to record the total coloring of K_k in matrix form as stated in the next Lemma.

Lemma 1. *Let $\{x_1, x_2, \ldots, x_k\}$ denote the vertices of K_k. There is a $k \times k$ matrix M_k obtained from a total coloring of K_k where*

$$m_{i,i} = color(x_i) \quad \text{and} \quad m_{i,j} = color(x_i, x_j) \quad \text{for } i \neq j$$

satisfying the following properties:

(a) the diagonal of M_k has the numbers $1, \ldots, k$ exactly once,

(b) if k is odd, then each row and column is (similarly) a permutation of the numbers $1, \ldots, k$,

(c) if k is even, then each row and column contains the numbers $1, \ldots, k+1$ at most once, and a different color among $1, \ldots, k$ is missing from each row/column.

As an example, the matrix M_9 is shown in Fig. 1. Matrix M_8 is obtained by deleting the bottom row and the rightmost column.

The total coloring for an odd-size clique will be used as a tool to color the edges of a rooted path graph after the following *Very Greedy Neighborhood Coloring* algorithm assigns colors to its vertices.

Very Greedy Neighborhood Coloring (VGNC) algorithm (for the vertices)

Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of a graph G, and let $N[v_i]$ denote the closed neighborhood of v_i. Let $k = \Delta(G) + 1$.

for $i = 1$ **to** n **do**

Assign distinct colors to all uncolored vertices in $N[v_i]$ from among
the unused colors in the current partial k-coloring of $N[v_i]$.
This is always possible since $k \geq degree(v_i) + 1$.

end

[1] In his monograph, Yap [16] cites Behzad, Chartrand and Cooper [2], but goes on to say that the construction was known "for hundreds of years" for edge colorings.

1	6	2	7	3	8	4	9	5
6	2	7	3	8	4	9	5	1
2	7	3	8	4	9	5	1	6
7	3	8	4	9	5	1	6	2
3	8	4	9	5	1	6	2	7
8	4	9	5	1	6	2	7	3
4	9	5	1	6	2	7	3	8
9	5	1	6	2	7	3	8	4
5	1	6	2	7	3	8	4	9

Fig. 1. The matrix M_9 totally coloring the clique K_9.

Warning: This assignment of colors may not be a proper coloring of $V(G)$, and even if it is a proper k-coloring, it may seem to be very wasteful using many more colors than is needed just to color the vertices. However, as a first phase for totally coloring rooted path graphs, it is exactly what we want.

Chordal graphs are precisely those which admit a perfect elimination ordering (PEO). The *reverse* of a PEO is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that for all $i < j < k$, if $v_i v_k, v_j v_k \in E(G)$ then $v_i v_j \in E(G)$. We will use the following new result that may also be useful for other subfamilies of chordal graphs.

Theorem 2. *If $v_1, v_2, \ldots, v_n$ is the reverse of a perfect elimination ordering, then the VGNC algorithm produces a proper vertex coloring.*

Proof. Let $v_j v_k \in E(G)$ with $j < k$. We will show that v_j and v_k are assigned different colors.

Denote by $step(v)$ the iteration during which vertex v received its color. Clearly, $step(v_t) \leq t$ for all t. Let $i = step(v_k)$. Since $v_k \in N[v_j]$, we have $i \leq j$, and clearly, $v_i v_k \in E(G)$ since $i = step(v_k)$.

If $i = j$, then v_k is assigned a different color from that of v_j by VGNC in iteration i.

Otherwise, $i < j < k$. In this case, since the considered ordering is the reverse of a perfect elimination ordering, we have $v_i v_j \in E(G)$. Thus, in iteration i, we have, by the VGNC, vertex v_k assigned a different color from that of v_j, since $v_j \in N[v_i]$. □

3. Totally coloring of rooted path graphs

A graph $G = (V, E)$ is a *rooted path graph* if there is a collection $\mathcal{P} = \{P_v | v \in V\}$ of directed paths in a rooted directed tree T associated to the vertices such that $uv \in E$ if and only if P_u and P_v intersect.

Lemma 2. *Let the paths of a representation $\mathcal{P}$ be ordered $\{P_1, P_2, \ldots, P_n\}$ "top–down" according to their roots, i.e., if r_i is above r_j in the tree, then $i < j$, where r_k denotes the root of path P_k. Then the following holds: If $i < j$ and $P_i \cap P_j \neq \emptyset$, then $r_j \in P_i \cap P_j$.*

Proof. Choose a point in $P_i \cap P_j$ and start walking towards the root of T. Since the paths are rooted and $i < j$, we will arrive at r_j while still in P_i. □

Our main result is the following: