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Some improved inequalities related to Vizing's conjecture

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ABSTRACT

Let $\gamma(G)$ be the domination number of a simple graph G and $G\Box H$ be the Cartesian product of two simple graphs G and H. A function $f: V(G) \to \{0,1,2\}$ is a Roman dominating function (RDF) if for each vertex $u \in V_0, N_G(u) \cap V_2 \neq \emptyset$, where $V_i = \{u \in V(G) : f(u) = i\}$. The Roman domination number $\gamma_R(G)$ is the minimum weight $f(V(G)) = \sum_{u\in V(G)} f(u)$ among all RDFs of G. Vizing conjectured in 1963 that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for any graphs G and H. To this day, this conjecture remains open. In this paper, we show that for each pair of simple graphs G and H, $\gamma(G\Box H) \geq \frac{1}{4}\gamma_R(G)\gamma_R(H)$. This means that Vizing's conjecture holds for any pair of Roman graphs G and H. Moreover, we prove $\gamma_R(G\Box H) \geq \gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$ if G or H is nonempty, which is a slight improvement of $\gamma_R(G\Box H) \geq \gamma(G)\gamma(H)$ obtained by Wu in 2013 [22].

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1. Introduction

In this paper, we refer the readers to [23] for undefined terminology and notation. Let $G = (V, E)$ be an undirected graph without loops, multi-edges and isolated vertices, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set, which is a subset of $\{xy|\ xy$ is an unordered pair of $V\}$. A graph G is *nonempty* if $E(G) \neq \emptyset$. Two vertices x and y are *adjacent* if $xy \in E(G)$. For graphs G and H, the Cartesian product $G\Box H$ is a graph with vertex set $V(G\Box H) = V(G) \times V(H)$ and two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. For a vertex x, let $N_G(x) = \{y : xy \in E(G)\}$ be the *open neighborhood* of x and let $N_G[x] = N(x) \cup \{x\}$ be the *closed neighborhood* of x. For a set $D \subseteq V(G)$, the *open neighborhood* of D is $N_G(D) = \bigcup_{u\in D}(N_G(u))$ and the *closed neighborhood* is $N_G[D] = N_G(D) \cup D$. Let $x \in D$. A vertex $y \in V(G) \setminus D$ is an *external private neighbor* of x with respect to D if $N_G(y) \cap D = \{x\}$. We use $G[D]$ to denote the subgraph of G induced by D. A set D of vertices is called *independent* if no two vertices in D are adjacent.

A set $D \subseteq V(G)$ is a *dominating set* of G if for any vertex $u \in V(G) - D$, $N_G(u) \cap D \neq \emptyset$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set is called a $\gamma(G)$*-set* if its cardinality is $\gamma(G)$.

Motivated by Stewart [18], a new variant of the domination number, the Roman domination number, was introduced by Cockayne et al. [9] in 2004. A function $f: V(G) \to \{0,1,2\}$ is a *Roman dominating function* (RDF) if for each vertex $u \in V_0$, $N_G(u) \cap V_2 \neq \emptyset$, where $V_i = \{u \in V(G) \mid f(u) = i\}$. The *weight* of f is given by $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight among all RDFs f of G. A RDF is a $\gamma_R(G)$*-function* if its weight is $\gamma_R(G)$. Note that there exists a 1-1 correspondence between the functions $f: V(G) \to \{0,1,2\}$ and (V_0, V_1, V_2). Thus, we write the function as $f = (V_0, V_1, V_2)$. It is well known that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ (as mentioned in [9]). A graph G is called a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$.

This definition of a Roman dominating function was given in [9]. We follow [9] to give another description of Roman dominating functions. A Roman dominating func-

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tion is a coloring of the vertices of a graph with the colors $\{0, 1, 2\}$ such that every vertex colored 0 is adjacent to at least one vertex colored 2. The definition of a Roman dominating function is given implicitly in [17,18]. The idea is that colors 1 and 2 represent either one or two Roman legions stationed at a given location (vertex v). A nearby location (an adjacent vertex u) is considered to be unsecured if no legions are stationed there (i.e. $f(u) = 0$). An unsecured location (u) can be secured by sending a legion to u from an adjacent location (v). The Emperor Constantine the Great, in the fourth century A.D., decreed that a legion cannot be sent from a location v if doing so leaves that location unsecured (i.e. if $f(v) = 1$). Thus, two legions must be stationed at a location $(f(v) = 2)$ before one of the legions can be sent to an adjacent location.

In 1963, Vizing [21] presented the following famous conjecture on the domination in Cartesian product.

Vizing's conjecture For any graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

This conjecture has received an increasing amount of attention in recent years. At this time, there are many relevant inequalities related to Vizing's conjecture [1,2,4–6,8, 11,12,15,19,20,22]. Many of the results related to the conjecture indicate that it holds for specific families of graphs or graphs satisfying a specific condition. One of the most successful approaches to this conjecture involves partitioning the vertex set of a graph G in a particular way. This approach was initiated by Barcalkin and German [2] in 1979, who proved that if $V(G)$ can be partitioned into $\gamma(G)$ sets each of which contains a clique, then G, called the BG-graph in [6], satisfies Vizing's conjecture. Whereafter, Hartnell and Rall [12] in 1995 stated that Vizing's conjecture holds for the class of Type χ graphs. Furthermore, Aharoni and Szabó [1] determined that chordal graphs satisfy Vizing's conjecture. The statement just mentioned, in fact, can be inferred by the result obtained by Brešar and Rall [5], who presented that Vizing's inequality is true for the graphs G with $\gamma_F(G) = \gamma(G)$, where $\gamma_F(G)$ is the fair domination number of G. Besides, the authors in [5] proved that the graphs G with $\gamma_F(G) = \gamma(G)$ present a generalization of the BG-graphs that is different from the class of Type χ graphs. Many well-known families of graphs, such as trees, cycles, the graphs with domination number 2, and the graphs having a 2-packing of cardinality equal to its domination number, are BG-graphs. Then the next two results are actually corollaries of the result in [2]. Jacobson and Kinch [15] proved $\gamma(G\Box T) \geq \gamma(G)\gamma(T)$ where T is a tree. El-Zahar and Pareek [11] derived that the conjecture holds when one of G and H is a cycle. Sun [20], and afterwards Brešar [4] with a new proof, stated that all graphs with domination number 3 satisfy Vizing's conjecture. This is the best contribution at present in terms of domination numbers of factors. In 2000, Clark and Suen [8] proved $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$ for any graphs G and H. Motivated by the above inequality, the bound to date for $\gamma(G\Box H)$ was improved to $\frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}min\{\gamma(G), \gamma(H)\}$ in 2012 by Suen and Tarr [19]. One of the few Vizing-like inequalities related to Roman domination number due to Wu [22], who presented $\gamma_R(G\Box H) \geq \gamma(G)\gamma(H)$ for any graphs G and H. Moreover, it is worth mentioning that the paper [6] surveyed all the contributions above and many other excellent achievements on Vizing's conjecture.

In this paper we obtain some results related to Vizing's conjecture. Section 2 shows a lower bound on the domination number of Cartesian product graphs. Vizing's conjecture is proved for two Roman graphs in this result. A lower bound on the Roman domination number of Cartesian product graphs is presented in Section 3. This inequality is a slight improvement of $\gamma_R(G\Box H) \geq \gamma(G)\gamma(H)$ which was obtained by Wu in 2013 [22].

2. Lower bound on domination number of Cartesian product graphs

First, we introduce two lemmas which are helpful for Theorem 2.3.

Lemma 2.1 *(Cockayne [9]). Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(G)$-function. Then*

(1) *No edge of G joins V_1 and V_2.*
(2) *V_2 is a $\gamma(G[V_0 \cup V_2])$-set.*

Lemma 2.2 *(Cockayne [9]). Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(G)$-function, where G is a graph without isolated vertices such that $\mid V_1 \mid$ is a minimum. Then*

(1) *V_1 is independent.*
(2) *$V_1 \subseteq N_G[V_0]$.*

Theorem 2.3. *For each pair of simple graphs G and H,*

$$\gamma(G\Box H) \geq \frac{1}{4}\gamma_R(G)\gamma_R(H).$$

Proof. Suppose that S_1, S_2 are the isolated vertex sets of graphs G and H, respectively. The proofs differ depending on $S_1 = S_2 = \emptyset$ and $S_1 \cup S_2 \neq \emptyset$. We will treat the two cases separately.

Case 1. $S_1 = S_2 = \emptyset$. Namely, both G and H are graphs without isolated vertices.

Let D be a $\gamma(G\Box H)$-set of the graph $G\Box H$ and $f = (B_0, B_1, B_2)$ be a $\gamma_R(G)$-function with $|B_1|$ minimum, where $B_1 = \{u_1, u_2, \ldots, u_{n_1}\}$ and $B_2 = \{u_{n_1+1}, u_{n_1+2}, \ldots, u_{n_1+n_2}\}$. Let $D_j = (B_j \times V(H)) \cap D$, where $j = 0, 1, 2$ and $D'_i = (\{u_i\} \times V(H)) \cap D$, where $i = 1, 2, \ldots, n_1 + n_2$. Then

$$\bigcup_{i=1}^{n_1} D'_i = D_1 \text{ and } \bigcup_{i=n_1+1}^{n_1+n_2} D'_i = D_2. \tag{2.1}$$

Denote by P_i the projection of D'_i onto H, that is,

$$P_i = \{v \in V(H) \mid (u_i, v) \in D'_i\},$$

where $i = 1, 2, \ldots, n_1 + n_2$. For each $i = 1, 2, \ldots, n_1 + n_2$, $g_i = (A_{0i}, A_{1i}, A_{2i})$ is a Roman domination function of H with $A_{0i} = N_H(P_i)$, $A_{1i} = V(H) - N_H[P_i]$ and $A_{2i} = P_i$. Therefore,

$$\gamma_R(H) \leq 2|A_{2i}| + |A_{1i}|. \tag{2.2}$$

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