Information Processing Letters 134 (2018) 14–17

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

An improved upper bound of edge–vertex domination number of a tree

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ARTICLE INFO

Article history:
Received 12 July 2016
Received in revised form 22 May 2017
Accepted 31 January 2018
Available online xxxx
Communicated by Jinhui Xu

Keywords:
Combinatorial problems
Total domination
Edge–vertex domination
Tree

ABSTRACT

An edge $e \in E(G)$ dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. An edge–vertex dominating set of a graph G is a set D of edges of G such that every vertex of G is edge–vertex dominated by an edge of D. The edge–vertex domination number of a graph G is the minimum cardinality of an edge–vertex dominating set of G. A subset $D \subset V(G)$ is a total dominating set of G if every vertex of G has a neighbor in D. The total domination number of G is the minimum cardinality of a total dominating set of G. We prove that for every nontrivial tree T of order n, with s support vertices we have $\gamma_{ev}(T) \le (\gamma_t(T) + s - 1)/2$, and we characterize the trees attaining this upper bound.

1. Introduction

Let $G = (V, E)$ be a simple graph. The open neighborhood of a vertex v of G is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$ and the set $N_G[v] = N_G(v) \cup \{v\}$ is called its closed neighborhood. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree one is a leaf and an edge incident with a leaf is called an end edge. The vertex adjacent to a leaf is called a support vertex. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that one of the components of $T - vx$ is a path P_n containing x as a leaf. (See Fig. 1.)

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A subset $D \subseteq V(G)$ is a total dominating set, abbreviated TDS, of G if every vertex of G has a neighbor in D. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. If D is a TDS of G of size $\gamma_t(G)$, then we call D a $\gamma_t(G)$-set. For more results on total domination, refer [1,2].

An edge $e \in E(G)$ dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. A subset $D \subseteq E(G)$ is an edge–vertex dominating set, abbreviated EVDS, of a graph G if every vertex of G is edge–vertex dominated by an edge of D. The edge–vertex domination number of a graph G, denoted by $\gamma_{ev}(G)$, is

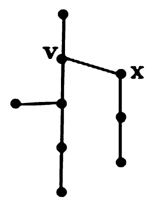

Fig. 1. A vertex v is adjacent to path P_3.

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https://doi.org/10.1016/j.ipl.2018.01.012

the minimum cardinality of an edge–vertex dominating set of G. If D is a EVDS of G of size $\gamma_{ev}(G)$, then we call D a $\gamma_{ev}(G)$-set. Edge–vertex domination in graphs was introduced in [3], and was further studied in [4].

Trees with total domination number equal to edge–vertex domination number plus one was characterized in [5]. We prove, for all trees of order n with s support vertices, $\gamma_{ev}(T) \le (\gamma_t(T)+s-1)/2$ and characterize trees attaining the bound. The bound obtained improves the upper bound of edge–vertex domination number in terms of total domination number as obtained in [5].

2. Results

Since the one-vertex graph does not have total dominating set and edge–vertex dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following straightforward observation.

Observation 1. *Every support vertex of a graph G is in every TDS of G.*

First we show that if T is a nontrivial tree of order n with s support vertices, then $\gamma_{ev}(T)$ is bounded above by $(\gamma_t(T)+s-1)/2$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_3, P_5\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations. (See Fig. 2.)

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to a support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertices to a vertex of T_k adjacent to path P_2.
- Operation $\mathcal{O}_3$: Attach a path P_4 by joining one of its leaves to a leaf adjacent to a weak support of T_k.

We now prove that for every tree T of the family $\mathcal{T}$, $\gamma_{ev}(T) = (\gamma_t(T)+s-1)/2$.

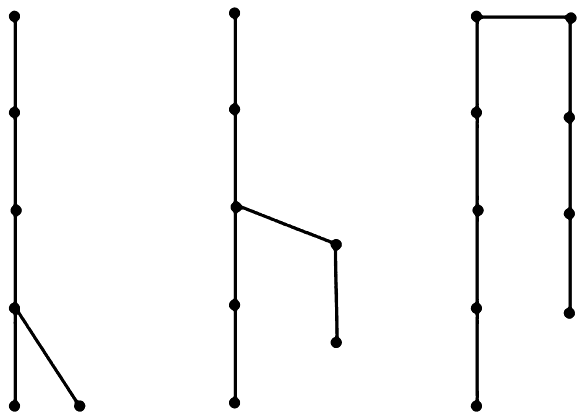

Fig. 2. Operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ performed on P_5.

Lemma 2. *If $T \in \mathcal{T}$, then $\gamma_{ev}(T) = (\gamma_t(T)+s-1)/2$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T_1 = P_3$, then $\gamma_{ev}(T_1) = 1$ and $\gamma_t(T_1) = 2$ or if $T_1 = P_5$, then $\gamma_{ev}(T_1) = 2$ and $\gamma_t(T_1) = 3$. It can be verified that $\gamma_{ev}(T_1) = (\gamma_t(T_1) + s-1)/2$ is satisfied. Let $k \ge 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let s' be the number of support vertices of the tree T'. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let y be the vertex joined to a support vertex x. Let z be a leaf adjacent to x. Let D' be a $\gamma_{ev}(T')$-set. Then x is dominated by some edge $xz \in D'$. If not so, the leaves of x would not be dominated by any edges of D'. The edge xz dominates y in the tree T. Thus D' is an EVDS of the tree T. We have $\gamma_{ev}(T) \le \gamma_{ev}(T')$. Let D be a $\gamma_{ev}(T)$-set. It is obvious that D is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \le \gamma_{ev}(T)$. We get $\gamma_{ev}(T) = \gamma_{ev}(T')$. Let D' be a $\gamma_t(T')$-set. By Observation 1, the support vertex $x \in D'$. The vertex x dominates y in the tree T. Thus D' is a TDS of the tree T. Thus $\gamma_t(T) \le \gamma_t(T')$. Let D be a $\gamma_t(T)$-set. It is obvious that D is a TDS of the tree T'. Thus $\gamma_t(T') \le \gamma_t(T)$. We get $\gamma_t(T') = \gamma_t(T)$. It is easy to see that $s = s'$. We now get $\gamma_{ev}(T) = \gamma_{ev}(T') = (\gamma_t(T') + s' - 1)/2 = (\gamma_t(T) + s - 1)/2$.

Assume that T is obtained from T' by operation $\mathcal{O}_2$. The vertex to which is attached P_2 we denote by x. Let u_1u_2 mean the attached path. Let u_1 be joined to x. The path P_2 adjacent to x and different from u_1u_2 we denote by v_1v_2. Let v_1 be adjacent to x. Let D' be a $\gamma_{ev}(T')$-set. The set $D' \cup \{xu_1\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \le \gamma_{ev}(T')+1$. Let D be a $\gamma_{ev}(T)$-set. To dominate the vertices v_2 and u_2, the edges $xv_1, xu_1 \in D$. It is clear that $D \setminus \{xu_1\}$ is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \le \gamma_{ev}(T) - 1$. We get $\gamma_{ev}(T) = \gamma_{ev}(T') + 1$. Let D' be a $\gamma_t(T')$-set. By Observation 1, the support vertex $v_1 \in D'$. To dominate the vertex v_1, the vertex $x \in D'$. The set $D' \cup \{u_1\}$ is a TDS of the tree T. Thus $\gamma_t(T) \le \gamma_t(T') + 1$. Let D be a $\gamma_t(T)$-set. By Observation 1, the vertices $u_1, v_1 \in D$. To dominate u_1 and v_1 the vertex $x \in D$. The set $D \setminus \{u_1\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \le \gamma_t(T) - 1$. We get $\gamma_t(T') = \gamma_t(T) - 1$. It is clear that $s' = s-1$. Thus $\gamma_{ev}(T) = \gamma_{ev}(T') + 1 = ((\gamma_t(T') + s' - 1)/2) + 1 = ((\gamma_t(T) - 1 + s - 1 - 1)/2) + 1 = (\gamma_t(T) + s - 1)/2$.

Assume that T is obtained from T' by operation $\mathcal{O}_3$. Let x be the leaf to which the path $P_4 = u_1u_2u_3u_4$ is attached. Let u_1 be joined to x. It is easy to see that $s' = s$. Let D' be a $\gamma_{ev}(T')$-set. The set $D' \cup \{u_2u_3\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \le \gamma_{ev}(T') + 1$. Let D be a $\gamma_{ev}(T)$-set. To dominate the vertex u_4, the edge $u_2u_3 \in D$. The set $D \setminus \{u_2u_3\}$ is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \le \gamma_{ev}(T) - 1$. We have $\gamma_{ev}(T) = \gamma_{ev}(T') + 1$. Let D be a $\gamma_t(T)$-set. By Observation 1, the vertex $u_3 \in D$. To dominate u_3, the vertex $u_2 \in D$. The set $D \setminus \{u_2, u_3\}$ is a TDS of the tree T'. Thus $\gamma_t(T') \le \gamma_t(T) - 2$. Let D' be a $\gamma_t(T')$-set. The set $D' \cup \{u_2, u_3\}$ is a TDS of the tree T. Thus $\gamma_t(T) \le \gamma_t(T') + 2$. We get $\gamma_t(T') = \gamma_t(T) - 2$. We now get $\gamma_{ev}(T) = \gamma_{ev}(T') + 1 = ((\gamma_t(T') + s' - 1)/2) + 1 = ((\gamma_t(T) - 2 + s - 1)/2) + 2 = (\gamma_t(T) + s - 1)/2$. □