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# Double Roman domination in trees ☆

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ABSTRACT

A subset $S$ of the vertex set of a graph $G$ is a dominating set if every vertex of $G$ not in $S$ has at least one neighbor in $S$. The domination number $\gamma(G)$ is defined to be the minimum cardinality among all dominating set of $G$.
A *Roman dominating function* on a graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function $f$ is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph $G$ is called the *Roman domination number* $\gamma_R(G)$ of $G$.
A *double Roman dominating function* on a graph $G$ is a function $f : V(G) \to \{0, 1, 2, 3\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 3$ or two vertices $v_1$ and $v_2$ for which $f(v_1) = f(v_2) = 2$, and every vertex $u$ for which $f(u) = 1$ is adjacent to at least one vertex $v$ for which $f(v) \geq 2$. The weight of a double Roman dominating function $f$ is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a double Roman dominating function on a graph $G$ is called the *double Roman domination number* $\gamma_{dR}(G)$ of $G$. Beeler et al. (2016) [6] showed that $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$ and showed that $2\gamma(T) + 1 \leq \gamma_{dR}(T) \leq 3\gamma(T)$ for any non-trivial tree $T$ and posed a problem that if it is possible to construct a polynomial algorithm for computing the value of $\gamma_{dR}(T)$ for any tree $T$. In this paper, we answer this problem by giving a linear time algorithm to compute the value of $\gamma_{dR}(T)$ for any tree $T$. Moreover, we give characterizations of trees with $2\gamma(T) + 1 = \gamma_{dR}(T)$ and $\gamma_{dR}(T) + 1 = 2\gamma_R(T)$.


## 1. Introduction

In this paper, we shall only consider graphs without multiple edges or loops. For notation and graph theory terminology we general follow [1]. Let $G$ be a graph, $v \in V(G)$, the *neighborhood* of $v$ in $G$ is denoted by $N(v)$. That is to say $N(v) = \{u | uv \in E(G), u \in V(G)\}$. The closed neighborhood $N[v]$ of $v$ in $G$ is defined as $N[v] = \{v\} \cup N(v)$. The *degree* of $v$ in $G$, denoted by $d_G(v)$, is the cardinality of its open neighborhood in $G$. The *distance* of two vertices $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest path between $u$ and $v$. A vertex of degree one is called a *leaf*. A graph is *trivial* if it has a single vertex. Denote by $P_n$ the path on $n$ vertices. Let $T$ be a tree. A vertex with exactly one neighbor is called a *leaf* and its neighbor is a *support vertex*. A support vertex with two or more leaf neighbors is called a *strong support vertex*.

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A set $S \subseteq V(G)$ in a graph $G$ is called a dominating set if $N[S] = V(G)$. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set in $G$. A *Roman dominating function* (RDF) on a graph $G$ is a function $f: V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function $f$ is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph $G$ is called the *Roman domination number* $\gamma_R(G)$ of $G$. Domination and Roman domination and their variations have received considerable attention [2–5]. As a variation of Roman domination, the concept of double Roman domination was proposed. A *double Roman dominating function* (DRDF) on a graph $G$ is a function $f: V(G) \to \{0, 1, 2, 3\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 3$ or two vertices $v_1$ and $v_2$ for which $f(v_1) = f(v_2) = 2$, and every vertex $u$ for which $f(u) = 1$ is adjacent to at least one vertex $v$ for which $f(v) \geq 2$. The weight $\omega(f)$ of a double Roman dominating function $f$ is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of a double Roman dominating function on a graph $G$ is called the *double Roman domination number* of $G$. We denote by $w_S(f)$ the weight of a double Roman dominating function $f$ in $S \subseteq V(G)$, i.e. $\omega_S(f) = \sum_{x \in S} f(x)$. We say that a function $f$ of $G$ is a $\gamma_{dR}$-function if it is a DRDF and $\omega(f) = \gamma_{dR}(G)$.

For an RDF $f$ of $G$, let $(V_0, V_1, V_2)$ be the ordered partition of $V(G)$ induce by $f$ such that $V_i = \{x : f(x) = i\}$ for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the function $f$ and the partition $(V_0, V_1, V_2)$ of $V$, we write $f = (V_0, V_1, V_2)$. Similarly, for a DRDF $f$ of $G$, let $(V_0, V_1, V_2, V_3)$ be the ordered partition of $V(G)$ induce by $f$ such that $V_i = \{x : f(x) = i\}$ for $i = 0, 1, 2, 3$. Note that there exists a 1-1 correspondence between the function $f$ and the partition $(V_0, V_1, V_2, V_3)$ of $V$, we write $f = (V_0, V_1, V_2, V_3)$.

**Lemma 1.** *[6] For any graph G, there exists a $\gamma_{dR}$-function of G such that no vertex needs to be assigned the value 1.*

Beeler et al. [6] showed that $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$ and defined a graph $G$ to be double Roman if $\gamma_{dR}(G) = 3\gamma(G)$. Moreover, they proposed the following problem.

**Problem 1.** Characterize the double Roman graphs. In particular, characterize the double Roman trees.

Especially, for any non-trivial tree $T$, Beeler et al. [6] proved that $2\gamma(T) + 1 \leq \gamma_{dR}(T) \leq 3\gamma(T)$ and every value in this range is realizable for trees.

**Lemma 2.** *[6] For any non-trivial tree $T$, $2\gamma(T) + 1 \leq \gamma_{dR}(T) \leq 3\gamma(T)$.*

Beeler et al. [6] also asked that if it is possible to construct a polynomial algorithm to compute the double Roman domination number of trees. In this paper, we give a linear algorithm to compute the double Roman domination number of trees. Moreover, we give a characterization of trees with $2\gamma(T) + 1 = \gamma_{dR}(T)$ and $\gamma_{dR}(T) + 1 = 2\gamma_R(T)$.

## 2. A characterization of trees $T$ with $\gamma_{dR}(T) = 2\gamma(T) + 1$ and $\gamma_{dR}(T) + 1 = 2\gamma_R(T)$

For any integers $r \geq 1$ and $t \geq 0$, let $F_{r,t}$ be a tree formed by joining $r$ edges and $t$ vertex-disjoint paths of length 2 as pendent paths to a single vertex $u$, which is called the *center* of $F_{r,t}$. Note that $F_{r,t}$ is a $(r+t)$-star with $t$ edges subdivided once.

We say $F_{r,t}$ is a *wounded spider* if $r \geq 1$ and $t \geq 0$ and $F_{r,t}$ is a *healthy spider* if $r = 0$ and $t \geq 2$. The *center vertex* of $F_{r,t}$ is also called the *head vertex* and the vertex at distance two from the head vertex is called the *foot vertex*. The vertex adjacent to the head vertex is called the *wounding vertex*.

**Theorem 1.** *For any integers $r \geq 1$ and $t \geq 0$, we have $\gamma_{dR}(F_{r,t}) = 2\gamma(F_{r,t}) + 1 = 2t + 3$.*

**Proof.** Let $V(F_{r,t}) = \{u, v_i, w_j : 1 \leq i \leq r + t, 1 \leq j \leq t\}$ and $E(F_{r,t}) = \{uv_i, v_jw_j : 1 \leq i \leq r + t, 1 \leq j \leq t\}$. Since any dominating set of $F_{r,t}$ contains at least one vertex of each edge in $\{uv_{t+1}, v_jw_j : 1 \leq j \leq t\}$, we have $\gamma(F_{r,t}) \geq t + 1$. On the other hand, since $r \geq 1$ and $\{u, w_j : 1 \leq j \leq t\}$ is a dominating set of $F_{r,t}$, we have $\gamma(F_{r,t}) \leq t + 1$. So $\gamma(F_{r,t}) = t + 1$.

Now we prove that $\gamma_{dR}(F_{r,t}) = 2t + 3$. Let $f$ be a DRDF of $F_{r,t}$. Then for any $j \in \{1, 2 \ldots, t\}$, $f(v_j) + f(w_j) \geq 2$ and equality holds if and only if $f(v_j) = 0$ and $f(w_j) = 2$. Moreover, $f(u) + f(v_{t+1}) \geq 2$ and equality holds if and only if $f(u) = 0$ and $f(v_{t+1}) = 2$. Since $f$ is a DRDF of $F_{r,t}$, we know that $f(v_j) + f(w_j) = 2$ and $f(u) + f(v_{t+1}) = 2$ do not appear simultaneously. Hence, $\omega(f) = \sum_{u \in V(G)} f(u) \geq 2t + 3$. On the other hand, let $g(u) = 3$, $g(v_i) = 0$ and $g(w_j) = 2$, where $i = 1, 2, \ldots, r + t$ and $j = 1, 2, \ldots, t$. It can be checked that $g$ is a DRDF of $F_{r,t}$ with weight $2t + 3$. Therefore, $\gamma_{dR}(F_{r,t}) = 2t + 3 = 2\gamma(F_{r,t}) + 1$. $\square$

**Theorem 2.** *Let $T$ be a nontrivial tree. Then $\gamma_{dR}(T) = 2\gamma(T) + 1$ if and only if $T$ is a wounded spider.*

**Proof.** The sufficiency follows from Theorem 1. Now we will prove that if $\gamma_{dR}(T) = 2\gamma(T) + 1$, there exist two integers $r \geq 1$ and $t \geq 0$ such that $T = F_{r,t}$. Suppose that this is not true and let $T$ be a smallest counterexample to the theorem. Then $\gamma_{dR}(T) = 2\gamma(T) + 1$ and $T \neq F_{r,t}$ for any integers $r \geq 1$ and $t \geq 0$.

**Claim 1.** *$T$ has no strong support vertex.*

**Proof of Claim 1.** Suppose that $T$ has a strong support vertex $x$. Let $y, z$ be two leaves of $T$ adjacent to $x$ and $T' = T - y$. Since $x$ is a support vertex in $T'$, we have $\gamma(T) = \gamma(T')$ and $\gamma_{dR}(T) \geq \gamma_{dR}(T')$. Hence, by Lemma 2, $\gamma_{dR}(T) \geq \gamma_{dR}(T') \geq 2\gamma(T') + 1 = 2\gamma(T) + 1$. Note that