Information Processing Letters 134 (2018) 47–51

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Intersection of the reflexive transitive closures of two rewrite relations induced by term rewriting systems

Sándor Vágvölgyi [1]

Department of Foundations of Computer Science, University of Szeged, Árpád tér 2, H-6720 Szeged, Hungary

ARTICLE INFO

Article history:
Received 29 April 2016
Received in revised form 12 September 2016
Accepted 4 February 2018
Available online 5 February 2018
Communicated by A. Tarlecki

Keywords:
Term rewriting systems
Post Correspondence Problem
Theory of computation

ABSTRACT

We show that it is undecidable whether the intersection of the reflexive transitive closures of two rewrite relations induced by term rewriting systems is equal to the reflexive transitive closure of a rewrite relation induced by a term rewriting system.

1. Introduction

In theoretical computer science, in particular in automated theorem proving and term rewriting, a binary relation $\to$ on a set of terms is called a rewrite relation if it is closed both under context application (the "replacement" or "monotonicity" property) and under substitutions (the "fully invariant property"), see Definition 1 in [2] and Definition 4.2.2 in [1]. The inverse, the symmetric closure, the reflexive closure, and the transitive closure of a rewrite relation are again rewrite relations [2]. The intersection of two rewrite relations is again a rewrite relation, and rewrite relations form a complete lattice with respect to intersection, see Section 2.2 in [2]. A term rewrite system (TRS for short) R induces a rewrite relation $\to_R$ [1]. A number of problems to characterise the intersection of various closures of rewrite relations induced by two TRSs have been considered in the literature [4,6,7].

By the above discussion, the intersection of the reflexive transitive closures of two rewrite relations induced by TRSs is also a rewrite relation. Hence it is natural to ask whether this intersection is equal to the reflexive transitive closure of a rewrite relation induced by a TRS. We show that the following problems are undecidable:

INSTANCE: Two convergent linear TRSs R and S on the same ranked alphabet Σ.

QUESTION: Does there exist a TRS U on Σ such that $\to_R^+ \cap \to_S^+ = \to_U^+$?

QUESTION: Does there exist a TRS U on Σ such that $\to_R^* \cap \to_S^* = \to_U^*$?

Here $\to_R^+$ and $\to_R^*$ denote the transitive closure and the reflexive transitive closure of $\to_R$, respectively.

2. Preliminaries

We present a review of the notions, notations and preliminary results used in the paper.

2.1. Abstract reduction systems

An abstract reduction system is a pair $(A, \to)$, where the reduction $\to$ is a binary relation on the set A. $\leftarrow$, $\leftrightarrow$, $\to^*$, and $\leftrightarrow^*$ denote the inverse, the symmetric closure, the reflexive transitive closure, and the reflexive transitive

E-mail address: vagvolgy@inf.u-szeged.hu.

[1] Research was supported by the *NKFI* grant No. K 108448.

https://doi.org/10.1016/j.ipl.2018.02.005
0020-0190/

symmetric closure of the binary relation $\rightarrow$, respectively. $x \in A$ is irreducible if there is no y such that $x \rightarrow y$. $y \in A$ is a normal form of $x \in A$ if $x \rightarrow^* y$ and y is irreducible. If $x \in A$ has a unique normal form, then it is denoted by $x\downarrow$. $y \in A$ is a descendant of $x \in A$ if $x \rightarrow^* y$.

The reduction $\rightarrow$ is called

- confluent if for all $x, y_1, y_2 \in A$, if $y_1 \leftarrow^* x \rightarrow^* y_2$, then $y_1 \rightarrow^* z \leftarrow^* y_2$ for some $z \in A$;
- terminating if there is no infinite chain $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$;
- convergent if it is both confluent and terminating.

2.2. Terms

$\mathbb{N}$ stands for the set of nonnegative integers, and $[1, n]$ stands for the set $\{1, \dots, n\}$ for each $n \in \mathbb{N}$. A ranked alphabet is a finite set Σ in which every symbol has a unique rank in $\mathbb{N}$. For each $m \in \mathbb{N}$, Σ_m denotes the set of all elements of Σ which have rank m. The elements of Σ_0 are called constants.

For a set of variables Y and a ranked alphabet Σ, $T_\Sigma(Y)$ denotes the set of Σ-terms (or Σ-trees) over Y, and $id_{T_\Sigma(Y)}$ denotes the identity relation on $T_\Sigma(Y)$. $T_\Sigma(\emptyset)$ is written as T_Σ. A term $t \in T_\Sigma$ is called a ground term. A term $t \in T_\Sigma(Y)$ is linear if each variable in Y occurs at most once in t. We specify a countable set $X = \{x_1, x_2, \dots\}$ of variables which will be kept fixed in this paper. Moreover, we put $X_m = \{x_1, \dots, x_m\}$ for $m \in \mathbb{N}$. Hence $X_0 = \emptyset$.

For a term $t \in T_\Sigma(X)$, the height $height(t)$ and the set of positions $POS(t) \subseteq N^*$ of t are defined by tree induction.

- If $t \in \Sigma_0 \cup X$, then $height(t) = 0$ and $POS(t) = \{\lambda\}$.
- If $t = f(t_1, \dots, t_m)$ with $f \in \Sigma_m$, $m > 0$, then $height(t) = 1 + \max\{height(t_i) \mid 1 \le i \le m\}$ and $POS(t) = \{i\alpha \mid 1 \le i \le m, \alpha \in POS(t_i)\}$.

For each $t \in T_\Sigma(X)$ and $\alpha \in POS(t)$, we introduce the subterm $t/\alpha \in T_\Sigma(X)$ of t at α and define the label $lab(t, \alpha) \in \Sigma \cup X$ in t at α as follows:

- for $t \in \Sigma_0 \cup X$, $t/\lambda = t$ and $lab(t, \lambda) = t$;
- for $t = f(t_1, \dots, t_m)$ with $m \ge 1$ and $f \in \Sigma_m$, if $\alpha = \lambda$ then $t/\alpha = t$ and $lab(t, \alpha) = f$, otherwise, if $\alpha = i\beta$ with $i \in [1, m]$, then $t/\alpha = t_i/\beta$ and $lab(t, \alpha) = lab(t_i, \beta)$.

For $t \in T_\Sigma$, $\alpha \in POS(t)$, and $r \in T_\Sigma$, we define $t[\alpha \leftarrow r] \in T_\Sigma$ as follows.

- If $\alpha = \lambda$, then $t[\alpha \leftarrow r] = r$.
- If $\alpha = i\beta$, for some $i \in N$ and $\beta \in N^*$, then $t = f(t_1, \dots, t_m)$ with $f \in \Sigma_m$ and $i \in [1, m]$. Then $t[\alpha \leftarrow r] = f(t_1, \dots, t_{i-1}, t_i[\beta \leftarrow r], t_{i+1}, \dots, t_m)$.

For a term $t \in T_\Sigma(X)$, $var(t)$ denotes the set of variables occurring in t, i.e. $var(t) = \{x_i \mid$ there exists $\alpha \in POS(t)$ such that $lab(t, \alpha) = x_i\}$.

For trees $t \in T_\Sigma(X_m)$, and $t_1, \dots, t_m \in T_\Sigma(X)$, we denote by $t[t_1, \dots, t_m]$ the tree obtained by substituting t_i for every occurrence of x_i in t, for each $i \in [1, m]$. We say that $t \in T_\Sigma(X_m)$ is a pattern of $s \in T_\Sigma(X)$, if there are $t_1, \dots, t_m \in T_\Sigma(X)$ such that $s = t[t_1, \dots, t_m]$.

For each $n \in \mathbb{N}$, an n-context over Σ is a term $u \in T_\Sigma(X_n)$ with exactly one occurrence of the variable x_i for each $i \in [1, n]$. $C_{\Sigma,n}$ denotes the set of n-contexts over Σ. Note that $C_{\Sigma,0} = T_\Sigma$. Let $C_\Sigma = \bigcup_{n \in \mathbb{N}} C_{\Sigma,n}$. We call a mapping $\omega : \Sigma \rightarrow \mathbb{N}$ a weight function. The weight function ω can be extended to a function $\omega : C_\Sigma \rightarrow \mathbb{N}$ as follows: let $\omega(u) = \sum_{f \in \Sigma} \omega(f) \cdot |u|_f$, where $|u|_f$ denotes the number of occurrences of symbol f in u. Thus, $\omega(u)$ simply adds up the weight of all occurrences of symbols of Σ in u.

An alphabet Δ is any finite nonempty set, Δ^* stands for the set of words over Δ, and λ denotes the empty word. For an alphabet Δ, we consider the ranked alphabet $\Delta \cup \{\#\}$, where $\# \notin \Delta$. Here each element of Δ is a unary symbol and $\#$ is a constant. Then we consider a tree in $T_{\Delta \cup \{\#\}}$ as a word over the alphabet $\Delta \cup \#$. For example, let $\Delta = \{a, b\}$. Then the tree $a(b(b(a(\#))))$ is written as the word $abba\#$. Conversely, for each word $w \in \Delta^*$, the word $w\#$ over the alphabet $\Delta \cup \{\#\}$ can be considered as a tree over the ranked alphabet $\Delta \cup \{\#\}$. For example, the word $aab\#$ can be considered as the tree $a(a(b(\#)))$.

For an alphabet Δ, we also consider the alphabets $\overline{\Delta} = \{\overline{a} \mid a \in \Delta\}$ and $\underline{\Delta} = \{\underline{a} \mid a \in \Delta\}$. The alphabets Δ, $\overline{\Delta}$, and $\underline{\Delta}$ are pairwise disjoint. For each word $w \in \Delta^*$, the word $\overline{w} \in \overline{\Delta}^*$ is defined as follows.

- If $w = \lambda$, then $\overline{w} = \lambda$.
- If $w = az$ for some $a \in \Sigma$ and $z \in \Delta^*$, then $\overline{w} = \overline{a}\,\overline{z}$.

For each word $w \in \Delta^*$, we define the word $\underline{w} \in \underline{\Delta}^*$ in a similar way to $\overline{w}$.

2.3. Term rewriting systems

Let Σ be a ranked alphabet. Then a term rewriting system (TRS) R on Σ is a finite subset of $(T_\Sigma(X) - X) \times T_\Sigma(X)$. For an element (l, r) of a TRS R, $var(r)$ is a subset of $var(l)$, and $l \notin X$. Elements (l, r) of R are called rules and are denoted by $l \rightarrow r$. The TRS R is linear if for each rule $l \rightarrow r$, both l and r are linear. A TRS R is context replacing if for each rule $l \rightarrow r$ of R, l and r are n-contexts for some $n \in \mathbb{N}$.

Let R be a TRS over Σ. For any terms $s, t \in T_\Sigma(X)$, position $\alpha \in POS(s)$, and rule $l \rightarrow r$ in R with $l, r \in T_\Sigma(X_m)$, $m \in \mathbb{N}$, we say that s rewrites to t applying the rule $l \rightarrow r$ at α, and denote this by $s \rightarrow_{l \rightarrow r, \alpha} t$ if there are $s_1, \dots, s_m \in T_\Sigma(X)$ such that $s/\alpha = l[s_1, \dots, s_m]$ and $t = s[\alpha \leftarrow r[s_1, \dots, s_m]]$. Here we also say that s rewrites to t and denote this by $s \rightarrow_R t$.

We say that a TRS R is confluent, terminating, or convergent, if $\rightarrow_R$ has the corresponding property. For a term $t \in T_\Sigma$, $R^*(t) = \{p \mid t \rightarrow_R^* p\}$ is the set of descendants of t, and $R^+(t) = \{p \mid t \rightarrow_R^+ p\}$ is the set of proper descendants of t.

Proposition 2.1. *Let TRS R be a context replacing TRS such that for each rule $l \rightarrow r$ in R, $\omega(l) > \omega(r)$. Then R is terminating.*

Proof. By direct inspection of the definitions. $\square$