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On f-colorings of nearly bipartite graphs ☆

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ABSTRACT

An f-coloring of a graph G is an edge coloring of G such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to f-color G is called the f-chromatic index of G and denoted by $\chi'_f(G)$. Any simple graph G has the f-chromatic index equal to $\Delta_f(G)$ or $\Delta_f(G)+1$, where $\Delta_f(G) = \max_{v\in V(G)}\{\lceil \frac{d(v)}{f(v)} \rceil\}$. If $\chi'_f(G) = \Delta_f(G)$, then G is of f-class 1, otherwise G is of f-class 2. In this paper, we give some sufficient conditions for a nearly bipartite graph to be of f-class 1.

1. Introduction

An *edge coloring* of a graph G is an assignment of some colors to all the edges of G. Let f be a function which assigns a positive integer $f_G(v)$ to each vertex $v \in V(G)$ in G. An *f-coloring* of G is an edge coloring of G such that each vertex $v \in V(G)$ has at most $f_G(v)$ adjacent edges colored with same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G, and denoted by $\chi'_f(G)$. If $f(v)=1$ for all $v \in V(G)$, then the f-coloring problem is reduced to the *proper edge coloring* problem.

f-colorings have interesting real-life applications in scheduling problems, e.g. the file transfer problem in computer network (see [6,7,10]). In this model, a vertex of graph G represents a computer, and an edge does a file which one wishes to transfer between the corresponding two computers. The integer $f(v)$ is the number of communication ports available at a computer (vertex) v. The edges having same color represent files that can be transferred simultaneously in the network. Thus, an f-coloring of G using $\chi'_f(G)$ colors corresponds to a scheduling of file transfers with the minimum completion time.

Since the proper edge-coloring problem is NP-complete even for regular graphs [9], the f-coloring problem is NP-complete also. Hakimi and Kariv [8] studied the f-coloring problem and obtained some upper bounds on $\chi'_f(G)$. Nakano et al. [12] obtained another upper bounds on $\chi'_f(G)$.

Our terminology and notations will be standard, except where indicated. Readers are referred to [4] for undefined terms. Throughout this paper, the *graph* refers to simple graph with a finite nonempty vertex set $V(G)$ and a finite nonempty edge set $E(G)$.

We define

$$\Delta(G) = \max_{v\in V(G)}\{d(v)\},$$

and

$$\Delta_f(G) = \max_{v\in V(G)}\{\lceil \frac{d(v)}{f(v)} \rceil\},$$

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in which $\lceil \frac{d(v)}{f(v)} \rceil$ is the smallest integer not smaller than $\frac{d(v)}{f(v)}$. So $1 \le \Delta_f(G) \le \Delta(G)$. $\Delta_f(G) = 1$ if and only if $d(v) \le f(v)$ for each $v \in V(G)$. If $\Delta_f(G) = 1$, then $\chi'_f(G) = 1$. In this paper, we just consider the non-trivial cases, i.e. the graphs with $2 \le \Delta_f(G) \le \Delta(G)$. It is easy to verify that $d(v) \le \Delta_f(G) f(v)$ for each $v \in V(G)$ and $\chi'_f(G) \ge \Delta_f(G)$. Hakimi and Kariv [8] generalized proper edge coloring to f-coloring and obtained the following result.

Theorem 1. *[8] Let G be a graph. Then*

$$\Delta_f(G) \le \chi'_f(G) \le \max_{v \in V(G)} \{\lceil \frac{d(v)+1}{f(v)} \rceil\} \le \Delta_f(G) + 1.$$

We say that a graph G is of *f-class* 1 if $\chi'_f(G) = \Delta_f(G)$, and G is of *f-class* 2 otherwise. The problem deciding whether a graph G is of f-class 1 or f-class 2 is called the *classification problem* on f-colorings. The problem attracts quite a lot of attention. Zhang and Liu [16–20], Zhang et al. [15,21,22], Yu et al. [14], Adivijaya et al. [1], Akbari et al. [2], Zhang and Zhu [24] and Cai et al. [5] studied some special kinds of graphs and obtained some sufficient conditions for a graph to be of f-class 1 or f-class 2. Liu et al. [11], Zhang et al. [23] and Akbari et al. [3] gave some properties of critical graphs on f-colorings.

When $f \equiv 1$, an f-coloring of a graph G is exactly an *edge coloring* of G, and the f-chromatic index of G is denoted by $\chi'(G)$ simply. If $f \equiv 1$, Theorem 1 is the famous theorem of Vizing [13].

Theorem 2. *[13] Let G be a graph. Then* $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$.

Similarly, we say a graph G is of *class* 1 if $\chi'(G) = \Delta(G)$, otherwise G is of *class* 2.

The number $d(v)/f(v)$ is called the *f-ratio* of vertex v in G. A vertex $u \in V(G)$ is *$\Delta_f(G)$-peelable*, if u has at most one remaining neighbor of f-ratio $\Delta_f(G)$.

Let $T \subset V(G)$. Let $G' = G \setminus T$ denote a graph obtained from G as follows:

$$\begin{cases} V(G') = V(G) \setminus T; \\ E(G') = E(G) \setminus \{uv : v \in T \text{ or } u \in T\}. \end{cases}$$

Zhang, Yan and Cai [23] obtained many interesting results, one of which will be used in this article as follows.

Lemma 3. *[23] Let G be a graph, $u \in V(G)$ and k be a positive integer with $k \ge \Delta_f(G)$. If u is $\Delta_f(G)$-peelable and $G \setminus \{u\}$ can be f-colored with k colors, then G can be f-colored with k colors.*

For the f-chromatic index of some special class of graphs, Hakimi and Kariv [8] obtained the following result.

Lemma 4. *[8] Let G be a bipartite graph. Then $\chi'_f(G) = \Delta_f(G)$.*

By Lemma 4, we can see that all bipartite graphs are of f-class 1. In this paper, we study the classification problem of nearly bipartite graphs and give some new sufficient conditions for a nearly bipartite graph to be of f-class 1.

2. Main results

In graph G, let

$$\begin{aligned} V_\Delta(G) &= \{v \in V(G) | d(v) = \Delta(G)\}, \\ V_{\Delta_f}(G) &= \{v \in V(G) | d(v) = \Delta_f(G) f(v)\}, \\ N^*(u) &= \{v \in N_G(u) | d(v) = \Delta(G)\}, \\ N^*_f(u) &= \{v \in N_G(u) | d(v) = \Delta_f(G) f(v)\}, \\ d^*(u) &= |N^*(u)| \text{ and } d^*_f(u) = |N^*_f(u)|. \end{aligned}$$

A graph G is called *$\Delta_f(G)$-peelable graph* if all the vertices of G can be iteratively peeled off in an order $v_1, v_2, ..., v_n$ using the following $\Delta_f(G)$-peeling operation: For each $1 \le i \le n$, peel off vertex v_i, if v_i has at most one neighbor v' satisfying that $v' \in V_{\Delta_f}(G)$ and $d_{G_{i-1}}(v') = d_G(v')$ in G_{i-1}, where $G_{i-1} = G \setminus \{v_1, v_2, ..., v_{i-1}\}$ $(2 \le i \le n)$ and $G_0 = G$.

Note that, for any subgraph G' of G, we define $f_{G'}(v) = f_G(v)$ for all $v \in V(G')$.

Theorem 5. *Let G be a graph with $\Delta_f(G) \ge 2$. Let H be a graph obtained from G by peeling off some vertices of G using the $\Delta_f(G)$-peeling operation. If H can be f-colored with $\Delta_f(G)$ colors, then G can be f-colored with $\Delta_f(G)$ colors.*

Proof. Without loss of generality, we can suppose that H is the graph obtained from G by peeling off vertices $v_1, v_2, ..., v_m$ of G in the order of $v_1, v_2, ..., v_m$ iteratively using the $\Delta_f(G)$-peeling operation. For $0 \le i \le m$, let $H_i = G \setminus \{v_1, v_2, ..., v_i\}$, where $H_0 = G$. It is easy to know $H = H_m$ and $H_{i+1} = H_i \setminus \{v_{i+1}\}$, $0 \le i \le m-1$. Since v_m is $\Delta_f(G)$-peelable and $H = H_m$ can be f-colored with $\Delta_f(G)$ colors, by Lemma 3, we know that H_{m-1} can be f-colored with $\Delta_f(G)$ colors. Using the same method, we can prove that H_i $(0 \le i \le m-2)$ can be f-colored with $\Delta_f(G)$ colors. Thus G can be f-colored with $\Delta_f(G)$ colors. □

A *nearly bipartite graph G* is a graph which is not bipartite but, for a vertex $u \in V(G)$, $G \setminus \{u\}$ is a bipartite graph with bipartition (X, Y), denoted by $G(X, Y; u)$.

In graph G, $u \in V(G)$, $N_G(u) = \{v_1, v_2, ..., v_{d(u)}\}$. Let $T = \{u_1, u_2, ..., u_t\}$, $T \cap V(G) = \emptyset$. Let $N_i \subset N_G(u)$, $1 \le i \le t$, $\bigcup_{1 \le i \le t} N_i = N_G(u)$ and $N_i \cap N_j = \emptyset$ for every $i, j \in \{1, 2, ..., t\}$, $i \ne j$. Construct an auxiliary graph G' from G as follows:

$$\begin{cases} V(G') = V(G) \setminus \{u\} \cup T; \\ E(G') = E(G) \setminus \{uv : v \in N_G(u)\} \\ \qquad \cup \{u_i v : v \in N_i, 1 \le i \le t\}. \end{cases} \tag{2.1}$$

G' is called *a splitting graph* of G.

In graph G, let $S = \{x_1, x_2, ..., x_s\} \subset V(G)$. *Identifying* $x_1, x_2, ..., x_s$ into x means that removal of the vertices in set S of G, adding a new vertex x to $G - S$ and joining x to each vertex in $N_G(S) \setminus S$ by an edge. The resulting graph is called *an identifying graph* of G.