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Path multicoloring in spider graphs with even color multiplicity ☆

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ABSTRACT

We give an exact polynomial-time algorithm for the problem of coloring a collection of paths defined on a spider graph using a minimum number of colors (MIN-PMC), while respecting a given even maximum admissible color multiplicity on each edge. This complements a previous result on the complexity of MIN-PMC in spider graphs, where it was shown that, for every odd k, the problem is NP-hard in spiders with admissible color multiplicity k on each edge. We also obtain an exact polynomial-time algorithm for maximizing the number of colored paths with a given number of colors (MAX-PMC) in spider graphs with even admissible color multiplicity on each edge.

1. Introduction

Path coloring problems have been studied extensively in the context of routing and wavelength assignment in optical networks, as well as in several other applications, including, for example, compiler optimization and vehicle scheduling. In a meaningful generalization, path multicoloring problems were defined and studied in [2,9,12,13,18]. Note that the term "path multicoloring" means that color collisions are allowed for edge-intersecting paths. This is in contrast to standard path coloring, where edge-intersecting paths must receive distinct colors.

Various optimization objectives have been studied in the context of path multicoloring. In this article, we are interested in two problems that arise from bounding the *admissible color multiplicity* on each edge, i.e., the maximum number of paths that can use this edge and receive the same color. In MIN-PMC, one seeks to color all paths with the minimum number of colors. The problem is defined formally as follows:

Problem 1 *(*MINIMUM PATH MULTICOLORING, MIN-PMC*).*
Instance: $\langle G, \mathcal{P}, \mu \rangle$, where $G = (V, E)$ is an undirected graph, $\mathcal{P}$ is a set of undirected simple paths on G, and $\mu : E \to \mathbb{N}$ is a function that maps each edge to its admissible color multiplicity.
Feasible solution: a coloring of $\mathcal{P}$ so that, for every edge e, at most $\mu(e)$ paths using e have the same color.
Goal: minimize the number of colors used.

In the maximization version of the problem, the number of available colors is limited and one seeks to maximize the number of paths that are colored. This problem, MAX-PMC, is defined formally as follows:

Problem 2 *(*MAXIMUM PATH MULTICOLORING, MAX-PMC*).*
Instance: $\langle G, \mathcal{P}, \mu, w \rangle$, where $G = (V, E)$ is an undirected graph, $\mathcal{P}$ is a set of undirected simple paths on G, $\mu : E \to \mathbb{N}$ is a function that maps each edge to its admissible

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color multiplicity, and $w \in \mathbb{N}$ is the number of available colors.
Feasible solution: a coloring of a subset $\mathcal{Q} \subseteq \mathcal{P}$ with at most w colors, so that, for every edge e, at most $\mu(e)$ paths using e have the same color.
Goal: maximize $|\mathcal{Q}|$.

The restriction of Min-PMC (resp. Max-PMC) to instances with $\mu(e) = 1$ for each edge e is known as the *minimum* (resp. *maximum*) *path coloring* problem and is denoted by Min-PC (resp. Max-PC).

Related work. Path coloring and path multicoloring problems have been extensively studied during the last twenty years (see e.g. [3–5,7–9,16–18,22] and references therein). Most meaningful variants are NP-hard in general graphs and even in simple topologies, e.g. stars and rings, whereas they can be solved exactly in chains. In addition, most variants are hard to approximate in general graphs within a constant factor. This, however, is possible in stars, rings, and in some other simple topologies.

The Min-PMC problem was first considered in [12,13] in the context of wavelength assignment in multifiber all-optical networks with an equal number of fibers per link. Complexity lower bounds and inapproximability results for Min-PMC follow from the corresponding results for Min-PC. Specifically, Min-PC is NP-hard already in rings and stars and one can exploit the connection of Min-PC in stars with the edge coloring problem in multigraphs to show that Min-PC in stars cannot have a $(\frac{4}{3} - \epsilon)$-approximation algorithm, unless P = NP [8,11]. Furthermore, there exists a constant $c > 0$ such that Min-PC in grids does not admit a $|V|^{\epsilon}$-approximation algorithm for any $\epsilon < c$, unless P = NP [15, Corollary 6.1].

Moving on to positive results for Min-PMC, it was shown in [6] that it admits a 4-approximation in trees. A $\frac{3}{2}$-approximation for stars was proposed in [16]; however, an asymptotic $\frac{9}{8}$-approximation can be obtained directly from the equivalence between Min-PMC in stars and f-coloring in multigraphs and the result of Nakano et al. for f-coloring of multigraphs [14]. In [16], the authors also give a 2-approximation algorithm for rings and an exact algorithm for chains. In [20] algorithms for Min-PMC in spiders[1] and caterpillars[2] were proposed, achieving approximation ratios of 2 and 3, respectively. In contrast, it was shown in [20] that the directed version of Min-PMC can be solved exactly in spiders.

A more recent result of Bian and Gu [5] states that Min-PMC can be solved exactly in spiders with *uniform* and even admissible color multiplicity (uniform stands for 'the same on each edge of the graph'). In fact, the same algorithm works under a slightly relaxed uniformity constraint, namely that each leg of the spider has uniform and even admissible color multiplicity, but the multiplicity may vary among different legs. Therefore, Min-PMC can also be solved exactly in stars where the admissible color multiplicity is even (not necessarily uniform). However, it is known that Min-PMC is NP-hard when G is a star and $\mu(e) = 1$ for all edges [8,11]. Bian and Gu also prove that, for every odd k, Min-PMC is NP-hard when G is a star and $\mu(e) = k$ for all edges.

Table 1
Known positive results for Min-PMC and Max-PMC in various topologies.

Topology	Best known approximation ratio for:			
	Min-PMC		Max-PMC	
rings	2	(cf. [16])	2	(cf. [21])
trees	4	(cf. [6])	2.54	(cf. [9])
caterpillars	3	(cf. [20])	2.54	(cf. [9])
spiders	2	(cf. [20])	1.58	(cf. [5])
spiders with $\mu(e)$ even and uniform over each leg	exact	(cf. [5])	exact	(cf. [5])
stars	3/2	(cf. [16])	1.58	(cf. [5])
	9/8 asympt.	(cf. [14])		
chains	exact	(cf. [16])	exact	(cf. [21])

Results for Max-PMC appear in [9], where a 2.54-approximation for trees is proposed, in [21], where the author gives an exact algorithm for chains and 2-approximation algorithms for rings and stars, and in [5], where they provide a 1.58-approximation algorithm for the problem in spiders and an exact algorithm for spiders with uniform and even admissible color multiplicity. The NP-hardness of Max-PMC in rings and stars follows from the NP-hardness of Max-PC in rings [19] and stars [7]. With regard to inapproximability, it is known that for some constant $c > 0$, Max-PC in grids does not admit a $|\mathcal{P}|^{\epsilon}$-approximation algorithm for any $\epsilon < c$, unless P = NP [15, Theorem 7.8], and that Max-PC in general graphs does not admit a $|E|^{\frac{1}{2}-\epsilon}$-approximation algorithm for any $\epsilon > 0$, unless NP $\subseteq$ ZPP [1].

We summarize the known positive results for Min-PMC and Max-PMC in various topologies in Table 1.

Our contributions. We show in Section 2 that Min-PMC can be solved exactly in spiders with non-uniform even admissible color multiplicity on each edge. This improves the result of Bian and Gu [5], by completely removing the uniformity requirement. Moreover, our result complements the complexity result in [5], where it was shown that allowing odd admissible color multiplicities in a star leads to NP-hardness of the problem.

As a corollary, we obtain in Section 3 an exact algorithm for Max-PMC in spiders with non-uniform even admissible color multiplicity. This result holds for any number of available colors.

Notation. Given an instance of Min-PMC or Max-PMC, we will use the notation L_e for the *load* of edge e with respect to the given path set, i.e., L_e is the number of paths that contain edge e.

2. Minimizing the number of colors

We solve Min-PMC in spiders with (non-uniform) even maximum color multiplicities. Let $\langle G, \mathcal{P}, \mu \rangle$ be an instance of Min-PMC, with $G = (V, E)$.

[1] A *spider* is a tree with at most one node of degree 3 or more.
[2] A *caterpillar* is a tree in which all nodes are within distance 1 of a central path.