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Independent feedback vertex sets for graphs of bounded diameter ☆

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ABSTRACT

The Near-Bipartiteness problem is that of deciding whether or not the vertices of a graph can be partitioned into sets A and B, where A is an independent set and B induces a forest. The set A in such a partition is said to be an independent feedback vertex set. Yang and Yuan proved that Near-Bipartiteness is polynomial-time solvable for graphs of diameter 2 and NP-complete for graphs of diameter 4. We show that Near-Bipartiteness is NP-complete for graphs of diameter 3, resolving their open problem. We also generalise their result for diameter 2 by proving that even the problem of computing a minimum independent feedback vertex is polynomial-time solvable for graphs of diameter 2.

1. Introduction

A graph is *near-bipartite* if its vertex set can be partitioned into sets A and B, where A is an independent set and B induces a forest. The set A is said to be an *independent feedback vertex set* and the pair (A, B) is said to be a *near-bipartite decomposition*. This leads to the following two related decision problems.

> Near-Bipartiteness
> *Instance:* a graph G.
> *Question:* is G near-bipartite (that is, does G have an independent feedback vertex set)?

> Independent Feedback Vertex Set
> *Instance:* a graph G and an integer $k \geq 0$.
> *Question:* does G have an independent feedback vertex set of size at most k?

Setting $k = n$ shows that the latter problem is more general than the first problem. Thus, if Near-Bipartiteness is NP-complete for some graph class, then so is Independent Feedback Vertex Set, and if Independent Feedback Vertex

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Set is polynomial-time solvable for some graph class, then so is Near-Bipartiteness.

Note that every near-bipartite graph is 3-colourable, that is, its vertices can be coloured with at most three colours such that no two adjacent vertices are coloured alike. The problems 3-Colouring [11] and Near-Bipartiteness [6] (and thus Independent Feedback Vertex Set) are NP-complete. However, their complexities do not necessarily coincide on special graph classes. Grötschel, Lovász and Schrijver [10] proved that Colouring is polynomial-time solvable for perfect graphs even if the permitted number of colours k is part of the input. However, Brandstädt et al. [5] proved that Near-Bipartiteness remains NP-complete for perfect graphs. The same authors also showed that Near-Bipartiteness is polynomial-time solvable for P_4-free graphs.

Yang and Yuan [16] proved that Near-Bipartiteness also remains NP-complete for graphs of maximum degree 4. To complement their hardness result, Yang and Yuan [16] showed that every connected graph of maximum degree at most 3 is near-bipartite except the complete graph K_4 on four vertices. This also follows from a more general result of Catlin and Lai [8]. Recently we gave a linear-time algorithm for finding an independent feedback vertex set in a graph of maximum degree at most 3 [4], and also proved that Near-Bipartiteness is NP-complete even for line graphs of maximum degree 4 [3]. It is also known that Near-Bipartiteness is NP-complete for planar graphs; this follows from a result of Dross, Montassier and Pinlou [9]; see the arXiv version of [4] for details.

Tamura, Ito and Zhou [15] proved that Independent Feedback Vertex Set is NP-complete for planar bipartite graphs of maximum degree 4 (note that Near-Bipartiteness is trivial for bipartite graphs). They also proved that Independent Feedback Vertex Set is linear-time solvable for graphs of bounded treewidth, chordal graphs and P_4-free graphs (the latter result generalising the result of [5] for Near-Bipartiteness on P_4-free graphs). In [3] we proved that finding a minimum independent feedback vertex set is polynomial-time solvable even for P_5-free graphs. We refer to [1,13] for FPT algorithms with parameter k for finding an independent feedback vertex set of size at most k.

The *distance* between two vertices u and v in a graph G is the length (number of edges) of a shortest path between u and v. The *diameter* of a graph G is the maximum distance between any two vertices in G. In addition to their results for graphs of bounded maximum degree, Yang and Yuan [16] proved that Near-Bipartiteness is polynomial-time solvable for graphs of diameter at most 2 and NP-complete for graphs of diameter at most 4. They asked the following question, which was also posed by Brandstädt et al. [5]:

What is the complexity of Near-Bipartiteness *for graphs of diameter 3?*

Our results. We complete the complexity classifications of Near-Bipartiteness and Independent Feedback Vertex Set for graphs of bounded diameter. In particular, we prove that Near-Bipartiteness is NP-complete for graphs of diameter 3, which answers the above question. We also prove that Independent Feedback Vertex Set is polynomial-time solvable for graphs of diameter 2. This generalises the result of Yang and Yuan [16] for Near-Bipartiteness restricted to graphs of diameter 2.

Theorem 1. *Let $k \geq 0$ be an integer.*

(i) *If $k \leq 2$, then* Independent Feedback Vertex Set *(and thus* Near-Bipartiteness*) is polynomial-time solvable for graphs of diameter k.*
(ii) *If $k \geq 3$, then* Near-Bipartiteness *(and thus* Independent Feedback Vertex Set*) is* NP*-complete for graphs of diameter k.*

We prove Theorem 1 (i) in Section 2. Yang and Yuan [16] proved their result for Near-Bipartiteness by giving a polynomial-time verifiable characterisation of the class of near-bipartite graphs of diameter 2. We use their characterisation as the starting point for our algorithm for Independent Feedback Vertex Set. In fact our algorithm not only solves the decision problem but even finds a minimum independent feedback vertex set in a graph of diameter 2.

We prove Theorem 1 (ii) in Section 3 by using a construction of Mertzios and Spirakis [12], which they used to prove that 3-Colouring is NP-complete for graphs of diameter 3. The outline of their proof is straightforward: a reduction from 3-Satisfiability that constructs, for any instance ϕ, a graph H_ϕ that is 3-colourable if and only if ϕ is satisfiable. We reduce 3-Satisfiability to Near-Bipartiteness for graphs of diameter 3 using the same construction, that is, we show that H_ϕ is near-bipartite if and only if ϕ is satisfiable. As such, our result is an observation about the proof of Mertzios and Spirakis, but, owing to the intricacy of H_ϕ, this observation is non-trivial to verify. In Section 3 we therefore repeat the construction and describe our reduction in detail, though we rely on [12] where possible in the proof.

2. Independent feedback vertex set for diameter 2

In this section we show how to compute a minimum independent feedback vertex set of a graph of diameter 2 in polynomial time. As mentioned, our proof relies on a known characterisation of near-bipartite graphs of diameter 2 [16]. In order to explain this characterisation, we first need to introduce some terminology.

Let $G = (V, E)$ be a graph and let $X \subseteq V$. Then the 2-*neighbour set* of X, denoted by A_X, is the set that consists of all vertices in $V \setminus X$ that have at least two neighbours in X. A set $I \subseteq V$ is *independent* if no two vertices of I are adjacent. For $u \in V$, we let $G - u$ denote the graph obtained from G after deleting the vertex u (and its incident edges). A graph is *complete bipartite* if its vertex set can be partitioned into two independent sets S and T such that there is an edge between every vertex of S and every vertex of T. If S or T has size 1, the graph is also called a *star*.