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Inefficiencies in network models: A graph-theoretic perspective

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ABSTRACT

We consider three network models where information items flow from a source to a sink node: flow networks, depletable channels, and traffic networks. We start with the standard model of flow networks; we characterise graph topologies that admit non-maximum saturating flows, under some capacity-to-edge assignment. We then consider a model where routing is constrained by energy available on nodes in finite supply (like in Smartdust) and efficiency is related to energy consumption and again to maximality of saturating flows. Finally, we consider a traffic model for selfish routing, where efficiency is related to latency at a Wardrop equilibrium. We show that all these forms of inefficiency yield different classes of graphs (apart from in the acyclic case, where the first and the last forms generate the same class). Interestingly, in all cases inefficient graphs can be made efficient by removing edges; this resembles a well-known phenomenon, called *Braess's paradox*.

1. Introduction

Through the years, several formal models have emerged for studying network design in terms of traffic, protocols, energy consumption, and so on (see, e.g., [1,5,16,18,22], just to cite a few). We are interested in networks as channels for transmitting information and aim at studying those net topologies where greedy routing always leads to optimal network utilisation. Thus, we model nets as st-digraphs, that are directed graphs with a chosen pair of nodes called *source* (s) and *sink* (t). At a given time, an amount of information is fed to the source and flows to the sink.

We start with the standard model of *flow networks* [1], where edges are endowed with capacities and a flow is possible only if it does not exceed the capacity of all edges it passes through. In the example of Fig. 1(b), two information units can flow from s to t, provided that one of them takes the 'northern' path $s\,u\,t$ and the other one takes the 'southern' path $s\,v\,t$. However, in a distributed scenario, due to partial knowledge of the net, the node u could choose to send to v the item received from s. In this case, the flow would take the path $s\,u\,v\,t$ and the net would be saturated by exchanging one information unit only. This can be considered a form of inefficiency, since not all possible flow is delivered. Indeed, flow networks built upon the graph W of Fig. 1(a), under some capacity-to-edge assignment (for example the one depicted in Fig. 1(b)), can have non-maximum saturating flows. We call *edge-weak* those graphs that, as W, suffer from this undesirable property. For calculating a maximum flow in an edge-weak graph, we cannot use a simple iterated DFS (or, equivalently, let every node autonomously choose nodes to forward information items it receives) but more sophisticated algorithms are needed (see, e.g., [1,2,13,21]).

Then, we consider *depletable channels* [7], introduced as a simple model for energy consumption in wireless

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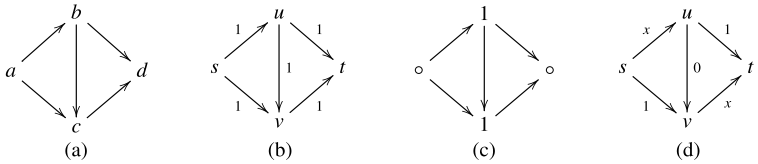

Fig. 1. The Wheatstone graph W (a) and three models built upon W: a flow network (b), a depletable channel (c), and a traffic network (d).

networks. There, st-digraphs have nodes equipped with a non-negative number representing depletable charge (as in Smartdust [24]). Devices have a depletable amount of energy that is consumed throughout their life because of information passing. By assuming that every information item transmitted consumes one charge unit of every node it passes through, charges place constraints on admissible flows. In Fig. 1(c), we depict a depletable channel built upon W, where nodes are labelled with their charge (1 for the nodes corresponding to b and c and '∘' – i.e., a big enough charge – for a and d). We can forward two information units, if we choose the 'right' paths (again, the northern and the southern ones), or one unit only, if we choose the 'wrong' path. Analogously to the case of flow networks, we call *node-weak* those graphs that exhibit this form of inefficiency, i.e. graphs that, for some charge-to-node assignment, admit non-maximum saturating flows.

Finally, we consider *traffic networks* [3,5], where edges are labelled with functions that model latency in terms of edge congestion, that in turn is modelled by the flow that passes through them. In this model, every flow is possible and the aim is to minimise the overall delay experienced in the system by autonomous and selfish users at the equilibrium of a noncooperative game, called *Wardrop equilibrium* [23]. In Fig. 1(d), we depict a traffic network on top of W, with two 'slow' edges ((u,t) and (s,v), that cause a delay of 1 independently of the amount of flow they receive), one 'ideal' edge ((u,v), where no delay is present) and two 'realistic' edges ((v,t) and (s,u), that cause a delay proportional to the amount of flow they receive). Selfish users, each of which controls a negligible part of traffic ε, choose the quickest path $s\,u\,v\,t$, because there they experience a delay of $2\,\varepsilon$ instead of $1+\varepsilon$ experienced in $s\,u\,t$ or $s\,v\,t$. However, this leads to the congestion of the path $s\,u\,v\,t$. If we consider a flow of value 1, at the Wardrop equilibrium all users pass along the path $s\,u\,v\,t$, and their experienced delay is 2. Paradoxically, if we remove the 'ideal' edge (u,v), at the Wardrop equilibrium, half of users will choose the path $s\,u\,t$ and the remaining half the path $s\,v\,t$: in such a case their experienced delay would be 3/2. This phenomenon has been known for a long time in the traffic network community as *Braess's paradox* [3,6], that occurs when the equilibrium cost may be reduced by removing an edge (or, equivalently, by raising the latency of such an edge). The property of a graph to lead, under some latency function, to the possibility of experiencing the Braess's paradox has been called *vulnerability* in [20].

Edge-weakness, node-weakness, and vulnerability share the characteristic that inefficient graphs can be amended by removing edges. In particular, if we remove the vertical edge in the examples given in Figs. 1(b), 1(c), and 1(d), we obtain networks that can be saturated only by maximum flows (in the first two cases) and that have the minimum delay at a Wardrop equilibrium (in the third case) under every capacity/charge/latency function, respectively.

Contributions. In this paper, we characterise all the inefficiencies described above for general directed st-graphs and compare them from a graph-theoretical perspective. In Section 2, we characterise *edge-weak* graphs. In Section 3, we recall the characterisation given in [8] for *node-weak* graphs. In Section 4 we recall the characterisation given in [9] for *vulnerable graphs* that extends similar characterisations for undirected graphs given in [17] and for a restricted family of directed graphs given in [10].

Stemming from these characterisations, in Section 5 we compare all these three classes of digraphs. In the general case, vulnerability implies edge-weakness; moreover, vulnerable graphs always contain an acyclic node-weak subgraph. This suggests that the core reason that makes a graph vulnerable is indeed its node-weak subgraphs. If we restrict our attention to DAGs, node-weakness implies both vulnerability and edge-weakness; these two classes coincide and they coincide with the class of graphs that are not series-parallel [19].

Interestingly, both in the acyclic and in the general case, the notions of node- and edge-weakness do *not* coincide. This is in sharp contrast with many classical results on flow networks, where models with capacities on nodes or on edges are interchangeable [1].

Preliminaries on graphs and flows. A directed graph $G=(V,E)$ consists of a set V of *vertices* (or *nodes*) and of a set $E \subseteq V \times V$ of *edges* (or *arcs*). Throughout the paper, we only consider *simple st-digraphs* that are directed graphs without self-loops and parallel edges, with a fixed source node s (without incoming edges) and sink node t (without outgoing edges). A (possibly cyclic) *path* p from a node u to v, notation $u \overset{p}{\leadsto} v$, is a sequence $z_1 \dots z_n$ of nodes such that for $1 \le i < n$, (z_i, z_{i+1}) is an edge, $z_1 = u$, and $z_n = v$. We say that p *touches* the node u if $u = z_i$ for some i ($1 \le i \le n$). Analogously, p *touches* the edge (u,v) if $u = z_i$ and $v = z_{i+1}$ for some i ($1 \le i < n$). Sometimes, we shall consider paths as sets of nodes (resp. of edges), and just write $u \in p$ (resp. $e \in p$) to denote that p touches node u (resp. edge e). Along the same way, if X is a set of nodes (resp. edges), we can write $p \cap X$ to denote the set of nodes (resp. edges) in X touched by p. An *st-path* is a path from s to t; we denote with $P(G)$ the set of all st-paths in G.

A *flow* φ is a function $\varphi : P(G) \to \mathbb{R}^+$. The *value* $|\varphi|$ of a flow is defined as $\sum_{p \in P(G)} \varphi(p)$. A flow induces a unique flow on edges and nodes. For any edge $e \in E$, $\varphi(e) =$