Information Processing Letters 115 (2015) 551–555

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

On Hamiltonian properties of unidirectional hypercubes

Chun-Nan Hung [a,*], Eddie Cheng [b], Tao-Ming Wang [c], Lih-Hsing Hsu [d]

[a] Department of Information Management, Da-Yeh University, Changhua, 51591, Taiwan, ROC
[b] Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, US
[c] Department of Applied Mathematics, Tunghai University, Taichung 40704, Taiwan, ROC
[d] Department of Computer Science and Information Engineering, Providence University, Taichung, 43301, Taiwan, ROC

ARTICLE INFO

Article history:
Received 20 November 2014
Accepted 21 January 2015
Available online 7 February 2015
Communicated by M. Yamashita

Keywords:
Interconnection networks
Unidirectional hypercubes
Fault Hamiltonicity

ABSTRACT

The study of fault Hamiltonicity is an important topic in studying the structures of interconnection networks. Indeed, many advanced results have been obtained for undirected interconnection networks. However, much less is known for the directed counterparts. In the note, we consider the fault Hamiltonicity for unidirectional hypercubes.

1. Introduction

Directed interconnection networks have been studied in the area of interconnection networks. Some research papers in this area include [1–6] and these references include many additional references. In particular, Ref. [4] gave an application and an architectural model for the studies of unidirectional graph topologies as well as a comparison of the diameters among many known unidirectional interconnection networks. In addition, Ref. [5] proposed unidirectional hypercubes as the basis for high speed networking. However, researchers know less about directed interconnection networks than their undirected counterparts. The main reason is that the directed version is usually more difficult. Fault Hamiltonicity has attracted a lot of attention in the area of interconnection networks. Rather than listing numerous references here, we invite the readers to [7] for its extensive reference list; moreover Chapters 11–13 contain many recent results. As an indication of why it is in general more difficult to study Hamiltonicity properties for directed graphs even for specified networks, we point out that although $C_m \times C_n$ is Hamiltonian, the directed version is only Hamiltonian for certain cases in which a necessary and sufficient condition is given in [8]. Although much is known about hypercubes and their various Hamiltonicity properties, as far as we know, there are no known corresponding results in the directed case. We hope this short note will interest researchers to divert some attention from undirected interconnection networks to their directed counterparts. We aim to write a concise note, so we forego most background materials and basic definitions.

Let $n \geq 2$. The *hypercube* Q_n has a vertex set consisting of 2^n binary strings of length n, where it is customary to denote a binary string of length n by $a = a_{n-1}a_{n-2}\dots a_1a_0$. Moreover, two vertices are adjacent if they differ in exactly one position, where the corresponding edge is an *i-edge* if the position that they differ is i. The *unidirectional hypercube* UQ_n is obtained by orienting the edges in Q_n. See [5]. Define $h(a_{n-1}a_{n-2}\dots a_1a_0) = a_{n-1} + a_{n-2} + \dots + a_1 + a_0$. Incidentally, this is called the *Hamming weight*. A vertex is called *even* if its Hamming weight is even; otherwise it is *odd*. Given an i-edge between u and v, then exactly one of $h(u)+i$ and $h(v)+i$ is even, say $h(u)+i$; we orient the edge from u to v. It is clear that if $h(u)$ is even

* Corresponding author.
E-mail addresses: spring@mail.dyu.edu.tw (C.-N. Hung), echeng@oakland.edu (E. Cheng), wang@thu.edu.tw (T.-M. Wang), lihhsing@gmail.com (L.-H. Hsu).

http://dx.doi.org/10.1016/j.ipl.2015.01.006
0020-0190/

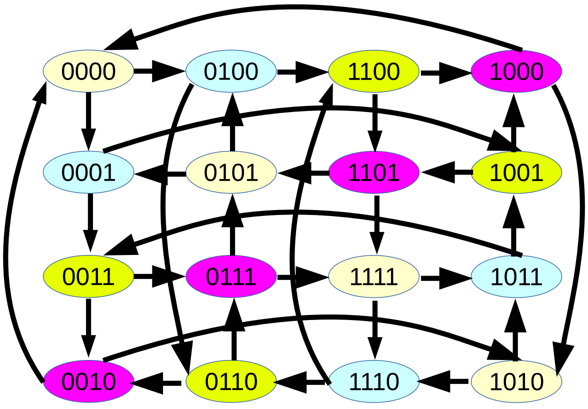

Fig. 1. UQ_4.

(respectively, odd), then $\rho(u) = \lfloor n/2 \rfloor$ and $\delta(u) = \lceil n/2 \rceil$ (respectively, $\delta(u) = \lfloor n/2 \rfloor$ and $\rho(u) = \lceil n/2 \rceil$), where $\rho(\cdot)$ and $\delta(\cdot)$ are the in-degree function and out-degree function, respectively. Moreover, we let $\rho(G)$ and $\delta(G)$ represent the minimum of $\rho(\cdot)$ and $\delta(\cdot)$ over all vertices of G, respectively. We note that UQ_2 is a directed 4-cycle. Fig. 1 gives UQ_4.

A directed graph is *Hamiltonian* if it contains a *(directed) Hamiltonian cycle*, that is, a (directed) cycle that contains every vertex of the graph. A *(directed) Hamiltonian path* in a graph is a (directed) path that contains every vertex of the graph. A directed graph $G = (V, E)$ is *k-arc-fault-tolerant-Hamiltonian* if for every $F \subseteq E$ with $|F| \le k$, $G - F$ is Hamiltonian. So a graph G is 0-arc-fault-tolerant-Hamiltonian corresponds to saying G is Hamiltonian. Clearly the best k that we can hope for is $\min\{\rho(G), \delta(G)\} - 1$. An even stronger property is to find a Hamiltonian cycle containing a prescribed arc. A directed graph G is *super Hamiltonian* if G has a Hamiltonian cycle containing e for every arc e in G. A directed graph $G = (V, E)$ is *super k-arc-fault-tolerant-Hamiltonian* if for every $F \subseteq E$ with $|F| \le k$, $G - F$ is super Hamiltonian. Thus G is super 0-arc-fault-tolerant-Hamiltonian is the same as saying G is super Hamiltonian. Given that a desirable property for an interconnection network is for it to be regular, it is more interesting to consider UQ_n when n is even. Clearly UQ_{2r} is r-regular. In the next section, we present our main result regarding these two fault Hamiltonian properties of UQ_{2r}.

2. Main result

As we have mentioned earlier, we are mainly interested in fault tolerant Hamiltonian results for regular graphs. We now present our main result.

Theorem 2.1. *UQ_2 is both Hamiltonian and super Hamiltonian. Let $r \ge 2$. Then UQ_{2r} is $(r-1)$-arc-fault-tolerant-Hamiltonian and super $(r-2)$-arc-fault-tolerant-Hamiltonian.*

Proof. The statement regarding UQ_2 is obviously correct. It is also easy to check that the result holds for UQ_4. We proceed with induction on r. Let $r \ge 3$. Let H_{ij} be the subgraph induced by the vertices of the form

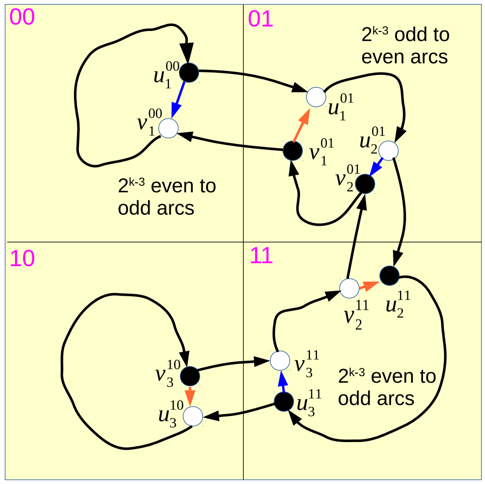

Fig. 2. Theorem 2.1.

$ija_{n-3} \dots a_1 a_0$. Then each H_{ij} is isomorphic to UQ_{2r-2}. So every H_{ij} is $(r-2)$-arc-fault-tolerant-Hamiltonian and super $(r-3)$-arc-fault-tolerant-Hamiltonian. Moreover, there are arcs between H_{ij} and H_{st} if and only if ij and st differ in exactly one position; in addition, there are 2^{2r-2} such arcs and exactly 2^{2r-3} of them are directed from H_{ij} to H_{st}. These arcs are called *cross arcs*. We use the notation u^{ij} to denote the vertex iju where each of i and j is a binary digit and u is a binary string of length $n-2$.

We first prove that UQ_{2r} is $(r-1)$-arc-fault-tolerant-Hamiltonian. Let F be a set of arcs in UQ_{2r} where $|F| \le r-1$. In fact, without loss of generality, we may assume that $|F| = r-1$. Let F_{ij} be the elements of F that are arcs in H_{ij} and let F_M be the elements of F that are cross arcs. Since Q_n is edge-transitive, we may assume that one of the elements in F is a cross arc. Thus $|F_{ij}| \le r-2$ for every ij. Hence every $H_{ij} - F_{ij}$ is Hamiltonian. Without loss of generality, we may assume that $|F_{00}| \ge |F_{01}|, |F_{11}|, |F_{10}|$. Thus $|F_{01}|, |F_{11}|, |F_{10}| \le r-3$. Let C_{00} be a directed Hamiltonian cycle in $H_{00} - F_{00}$. There are 2^{2r-3} arcs on C_{00} of the form (u_1^{00}, v_1^{00}) where u_1^{00} is even and hence v_1^{00} is odd. Such an arc is good if both cross arcs (u_1^{00}, u_1^{01}) and (v_1^{01}, v_1^{00}) are not in F. We note that these cross arcs are oriented in the correct way as u_1 is even and their endpoints differ in the $(2r-2)$th position. Since $|F_M| \le r-1$ and $2^{2r-3} > r-1$ as $r \ge 3$, such a good arc can be found. Now consider H_{01}, we have the arc (v_1^{01}, u_1^{01}). (We note that the orientation is correct as u_1^{01} is odd.) This arc may or may not be in F_{01}. (This arc will not be used in our construction of a desired Hamiltonian cycle.) In any case, we let $F'_{01} = F_{01} - \{(v_1^{01}, u_1^{01})\}$. Thus $|F'_{01}| \le r-3$ and by the induction hypothesis, there is a Hamiltonian cycle C_{01} in $H_{01} - F'_{01}$ containing (v_1^{01}, u_1^{01}). By the same argument, we can find a good arc (u_2^{01}, v_2^{01}) on C_{01} where u_2^{01} is odd, so the cross arcs (u_2^{01}, u_2^{11}) and (v_2^{11}, v_2^{01}) are not elements of F. (Again the orientation of the arcs are correct as u_2^{01} is odd, and u_2^{01} and u_2^{11} differ in the $(2r-1)$th position.) It is now clear how to complete the proof as shown in Fig. 2. (A black vertex is an even vertex and a white vertex is an odd vertex.)