Information Processing Letters 115 (2015) 604–608

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Alternative formulations for the ordered weighted averaging objective ☆

André Chassein, Marc Goerigk *

Fachbereich Mathematik, Technische Universität Kaiserslautern, Germany

ARTICLE INFO

Article history:
Received 15 October 2014
Received in revised form 10 February 2015
Accepted 10 February 2015
Available online 14 February 2015
Communicated by B. Doerr

Keywords:
Combinatorial problems
Multi-criteria optimization
Ordered weighted averaging
Linear programming

ABSTRACT

The ordered weighted averaging (OWA) objective is an aggregate function over multiple optimization criteria that has received increasing attention by the research community over the last decade. Different to the weighted sum, where a certain weight is assigned to every objective function, weights are attached to ordered objective functions (i.e., for a fixed solution, objective functions are sorted with respect to their size, and weights are assigned to positions within this ordering). As this contains max–min or worst-case optimization as a special case, OWA can also be considered as an alternative approach to robust optimization.

For linear programs with OWA objective, compact and extended reformulations exist. We present new such reformulation models with reduced size. A computational comparison indicates that these formulations improve solution times.

1. Introduction

We consider multi-criteria optimization problems of the form

$$\max \left\{ Cx \mid x \in \mathcal{X} \right\}, \tag{1}$$

where $C \in \mathbb{R}^{k \times n}$ is a matrix of linear objective functions and $\mathcal{X} \subseteq \mathbb{R}^n$ denotes some set of feasible solutions; e.g., for linear programs, we have

$$\mathcal{X} = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$$

for a coefficient matrix $A \in \mathbb{R}^{m \times n}$ and a right-hand size $b \in \mathbb{R}^m$. As multiple objectives have to be considered simultaneously, different approaches to what an "optimal" solution constitutes have been proposed in the literature (for an overview, see [2]). In this paper, we follow the ordered weighted averaging (OWA) approach as introduced by Yager [11], which aggregates the multi-criteria problem to a single-criterion counterpart.

To formulate the OWA aggregate function, we consider the ordering map $\Theta : \mathbb{R}^k \to \mathbb{R}^k$ with $\Theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_k(y))$ that permutes the vector components of y such that $\theta_i(y) \leq \theta_{i+1}(y)$ for $i = 1, \ldots, k-1$. Given a weight vector $w \in \mathbb{R}^k$, the OWA problem is then defined as

$$\max \left\{ \sum_{i \in [k]} w_i \theta_i(Cx) \mid x \in \mathcal{X} \right\}, \tag{2}$$

where we use the notation $[k] := \{1, \ldots, k\}$. Note that the OWA concept can be used to generalize classic aggregation functions in multiple criteria optimization, as, for example, the average case or worst case objective. But even more interesting is the fact that different intermediate objectives between the average and worst case objective functions can be defined using appropriate OWA weights.

☆ Effort sponsored by the Air Force Office of Scientific Research, Air Force Material Command, USAF, under grant number FA8655-13-1-3066. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purpose notwithstanding any copyright notation thereon.

* Corresponding author.
E-mail address: goerigk@mathematik.uni-kl.de (M. Goerigk).

http://dx.doi.org/10.1016/j.ipl.2015.02.008

As frequently done, we make the assumptions of *equitability*, which is given if $w_1 > w_2 > \ldots > w_k > 0$ (see also [5]). In [6], this case is considered for linear programs, and reformulations of (2) are presented. They show that the problem is equivalent to

$$\max\ z \tag{3}$$
$$\text{s.t.}\quad Cx = y \tag{4}$$
$$z \le \sum_{i\in[k]} w_{\tau(i)} y_i \qquad \forall \tau \in \Pi \tag{5}$$
$$Ax = b \tag{6}$$
$$x \in \mathbb{R}^n_+,\ y \in \mathbb{R}^k,\ z \in \mathbb{R}, \tag{7}$$

where Π denotes all permutations of $[k]$. Dualizing this problem, one can use column generation over the dual variables associated with the permutations. Furthermore, they present the following compact model:

$$\max\ \sum_{j\in[k]} j w'_j r_j - \sum_{i\in[k]}\sum_{j\in[k]} w'_j d_{ij} \tag{8}$$
$$\text{s.t.}\quad Cx = y \tag{9}$$
$$d_{ij} \ge r_j - y_i \qquad \forall i, j \in [k] \tag{10}$$
$$Ax = b \tag{11}$$
$$x \in \mathbb{R}^n_+,\ d \in \mathbb{R}^{k\times k}_+,\ y \in \mathbb{R}^k,\ r \in \mathbb{R}^k, \tag{12}$$

where $w'_j = w_j - w_{j+1}$ for all $j = 1, \ldots, k-1$ and $w'_k = w_k$. A model of this type has been applied, e.g., to the multi-objective spanning tree problem in [3], and to facility location problems in [4]. In the following we present a new model that can be useful to any of these applications.

2. Alternative models

We present different approaches to reformulate problem (2). Starting from the observation that

$$\sum_{i\in[k]} w_i \theta_i(Cx) = \min_{\tau\in\Pi} \sum_{i\in[k]} w_{\tau(i)} \theta_i(Cx)$$

as presented in [6], we reconsider the model

$$\max\ \min_{\tau\in\Pi} \sum_{i\in[k]} w_{\tau(i)} y_i \tag{13}$$
$$\text{s.t.}\quad Cx = y \tag{14}$$
$$x \in \mathcal{X}. \tag{15}$$

To reformulate the problem we introduce the polytope of permutation matrices P^Π (see [1] or [10, Chapter 18]). The linear description of P^Π is given by

$$P^\Pi = \left\{ p \in \mathbb{R}^{k\times k}_+ \;\middle|\; \sum_{i\in[k]} p_{ij} = 1\ \forall j \in [k],\ \sum_{j\in[k]} p_{ij} = 1\ \forall i \in [k] \right\}.$$

An arbitrary permutation $\tilde{\tau} \in \Pi$ can represented by the point $\tilde{p} \in \mathbb{R}^{n\times n}_+$, where

$$\tilde{p}_{ij} = \begin{cases} 1, \text{if } \tilde{\tau}(i) = j \\ 0, \text{else.} \end{cases}$$

Note that all vertices of P^Π are integral and that there is a one-to-one correspondence between vertices of P^Π and permutations in Π. Instead of considering all permutations in Π, one can also relax the objective function (13) and use the convex hull of permutations, i.e., P^Π. This does not change the objective value of the inner minimum, as there is always a vertex of P^Π at which the optimum is attained. Therefore, we can rewrite the inner optimization problem

$$\min_{\tau\in\Pi} \sum_{i\in[k]} w_{\tau(i)} y_i$$

for fixed values of y as

$$\min\ \sum_{i\in[k]}\sum_{j\in[k]} w_j y_i p_{ij} \tag{16}$$
$$\text{s.t.}\quad \sum_{i\in[k]} p_{ij} = 1 \qquad \forall j \in [k] \tag{17}$$
$$\sum_{j\in[k]} p_{ij} = 1 \qquad \forall i \in [k] \tag{18}$$
$$p \in \mathbb{R}^{k\times k}_+ \tag{19}$$

the dual of which is

$$\max\ \sum_{i\in[k]} (\alpha_i + \beta_i) \tag{20}$$
$$\text{s.t.}\quad \alpha_i + \beta_j \le w_j y_i \qquad \forall i, j \in [k] \tag{21}$$
$$\alpha, \beta \in \mathbb{R}^k. \tag{22}$$

Using this reformulation of the inner problem, we get the following new compact formulation for problem (2):

$$\max\ \sum_{i\in[k]} (\alpha_i + \beta_i) \tag{23}$$
$$\text{s.t.}\quad Cx = y \tag{24}$$
$$\alpha_i + \beta_j \le w_i y_j \qquad \forall i, j \in [k] \tag{25}$$
$$x \in \mathcal{X} \tag{26}$$
$$\alpha, \beta, y \in \mathbb{R}^k. \tag{27}$$

This formulation needs $3k$ additional variables compared to the original problem (1), and $k^2 + k$ new constraints. In comparison, model (8)–(12) requires $k^2 + 2k$ additional variables and $k^2 + k$ new constraints. Thus, the formulation we propose needs an order of k variables less.

One might also use an extended description of P^Π to do column generation in the primal problem (note that in [6], column generation needed the dual). To this end, we write the inner optimization problem as follows:

$$\min\ \sum_{i\in[k]} y_i p_i \tag{28}$$
$$\text{s.t.}\quad \sum_{i\in S} p_i \ge \varphi_{|S|} \qquad \forall\, \emptyset \subsetneq S \subsetneq [k] \tag{29}$$