Information Processing Letters 114 (2014) 31–37

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

# Motzkin subposets and Motzkin geodesics in Tamari lattices

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ARTICLE INFO

*Article history:*
Received 8 May 2013
Received in revised form 26 August 2013
Accepted 3 October 2013
Available online 10 October 2013
Communicated by M. Yamashita

*Keywords:*
Combinatorial problems
Lattices
Dyck words
Motzkin words
Tamari lattice
Metric
Geodesic

ABSTRACT

The Tamari lattice of order $n$ can be defined by the set $\mathcal{D}_n$ of Dyck words endowed with the partial order relation induced by the well-known rotation transformation. In this paper, we study this rotation on the restricted set of Motzkin words. An upper semimodular join semilattice is obtained and a shortest path metric can be defined. We compute the corresponding distance between two Motzkin words in this structure. This distance can also be interpreted as the length of a geodesic between these Motzkin words in a Tamari lattice. So, a new upper bound is obtained for the classical rotation distance between two Motzkin words in a Tamari lattice. For some specific pairs of Motzkin words, this bound is exactly the value of the rotation distance in a Tamari lattice. Finally, enumerating results are given for join and meet irreducible elements, minimal elements and coverings.



## 1. Introduction and notations

The set $\mathcal{D}$ of Dyck words over $\{(,)\}$ is the language defined by the grammar $S \rightsquigarrow \lambda|(S)|SS$ where $\lambda$ is the empty word, *i.e.* the set of well-formed parentheses strings. Let $\mathcal{D}_n$ be the set of Dyck words of length $2n$, *i.e.* with $n$ open and $n$ close parentheses. The cardinality of $\mathcal{D}_n$ is the $n$th Catalan number $c_n = (2n)!/(n!(n+1)!)$ (see A000108 in [23]). For instance, $\mathcal{D}_3$ consists of the five words ()()(), (())(), ()(()), (()()) and ((())). A large number of various classes of combinatorial objects are enumerated by the Catalan sequence. This is the case, among others, for ballot sequences, planar trees, binary rooted trees, nonassociative products, stack sortable permutations, triangulations of polygons, and Dyck paths. See [24] for a compilation of such Catalan sets. Some of them are endowed with a partial ordering relation [1–3,10,20,21]. For instance, the coverings of the so-called Tamari lattices [11,13,14,16,18,22,25] can be defined by different elementary transformations depending on the Catalan set considered. The most known is the semi-associative law $x(yz) \longrightarrow (xy)z$ for well-formed parenthesized expressions involving $n$ variables. Also, the Tamari lattice of order $n$ can be defined on the set $\mathcal{T}_n$ of binary rooted trees with $n+1$ leaves. Indeed, from a well-formed parenthesized expression on $n$ variables, we consider the bijection that recursively constructs the binary rooted tree where the left (resp. right) subtree is defined by the left (resp. right) part of the expression. For example, the binary rooted trees associated to the two expressions $x(yz)$ and $(xy)z$ are illustrated in Fig. 1. Moreover, the semi-associative law on parenthesized expressions is equivalent to the well-known left-rotation on binary trees showed in Fig. 1.

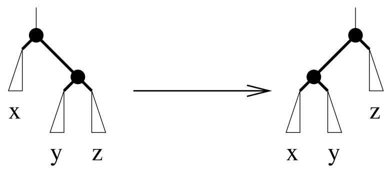


**Fig. 1.** The left-rotation transformation on binary trees.

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0020-0190/$ – see front matter 
http://dx.doi.org/10.1016/j.ipl.2013.10.001

Now, let $T \in \mathcal{T}_n$ be a binary tree with $n+1$ leaves. Reading $T$ in prefix order and replacing each internal node (resp. each leaf except the last) with an open (resp. a close) parenthesis, we obtain a bijection from $\mathcal{T}_n$ to $\mathcal{D}_n$ that translates the left-rotation transformation into the elementary transformation $(u)(\longrightarrow ((u)$ where $u$ is a Dyck word. This transformation will be also called left-rotation. More precisely, we say that $d' \in \mathcal{D}_n$ is obtained from $d \in \mathcal{D}_n$ by a left-rotation if $d = \alpha(u)(\beta$ and $d' = \alpha((u)\beta$ where $u$ is a Dyck word and $\alpha$ (resp. $\beta$) is some prefix (resp. suffix) of some Dyck word. The inverse transformation is called right-rotation. For instance, (())**((**()(())()**)**()) is obtained from (())**(**()(())() **)(**()) by a left-rotation.

We define the rotation distance between two Dyck words as the minimum number of left- and right-rotations necessary to transform one word into the other. There remains today an open problem whether the rotation distance can be computed in polynomial time. Previous works on rotation distance have focused on approximation algorithms [4,7,18,19].

In this paper, we study this rotation on the restricted set $\mathcal{M}$ of Motzkin words defined by the grammar $S \rightsquigarrow \lambda|(SS)$. In Section 2, an upper semimodular join semilattice is constructed. In this structure, we compute the length of a shortest path between two Motzkin words. This distance can also be interpreted as the length of a geodesic between these two Motzkin words in a Tamari lattice, *i.e.* the length of a shortest path between them by browsing through Motzkin words only. So, a new upper bound is obtained for the classical rotation distance between two Motzkin words in a Tamari lattice. For some specific pairs of Motzkin words, this bound is exactly the value of the well-known rotation distance in a Tamari lattice. In Section 3, enumerating results are given for join and meet irreducible elements, minimal elements and coverings.

## 2. Motzkin geodesics

Let $\mathcal{M}$ be the set of Motzkin words, *i.e.* the language over $\{(,)\}$ defined by the grammar $S \rightsquigarrow \lambda|(SS)$. Let $\mathcal{M}_n$ be the set of Motzkin words of length $2n$, *i.e.* with $n$ open and $n$ close parentheses. The cardinality of $\mathcal{M}_n$ is the $n$th term of the Motzkin sequence A001006 in [23]. For example, $\mathcal{M}_4 = \{((((())))), ((()())), (()(())), ((())())\}$. Obviously we have $\mathcal{M}_n \subseteq \mathcal{D}_n$ for $n \geqslant 1$. We refer to [8] and [24] for several combinatorial classes enumerated by the Motzkin numbers.

The following lemma shows how the left-rotation between two Motzkin words can be expressed by a natural transformation $\longrightarrow$ in $\mathcal{M}_n$.

**Lemma 1.** *Let $m, m' \in \mathcal{M}_n$. Then, the word $m'$ is obtained from $m$ by a left-rotation if and only if $m'$ is obtained from $m$ by a left-transformation $u(v) \longrightarrow (uv)$ where $u$ and $v$ are two Motzkin words, i.e. $m = \alpha u(v)\beta$ and $m' = \alpha(uv)\beta$ where $u, v \in \mathcal{M}$ and $\alpha$ (resp. $\beta$) is some prefix (resp. suffix) of some Motzkin word.*

**Proof.** Let $m$ and $m'$ be two Motzkin words such that $m'$ is obtained from $m$ by a left-rotation. Thus, there is a Dyck word $u$ such that $m = \alpha(u)(\beta$ and $m' = \alpha((u)\beta$ where $\alpha$ and $\beta$ are some prefix and some suffix of $m$ and $m'$. Since $m'$ is a Motzkin word, it necessarily is of the form $m' = \alpha((u)v)\beta'$ where $(u)$ and $v$ are Motzkin words with $\beta = v)\beta'$. We deduce that $m = \alpha(u)(v)\beta'$. Furthermore, the fact that $(v)$ and $(u)$ are Motzkin words necessarily implies that $m$ is of the form $m = \alpha'((u)(v))\beta''$ and thus, $m' = \alpha'(((u)v))\beta''$ for some $\alpha'$ and $\beta''$. Thus the left-rotation between two Motzkin words is equivalent to the transformation $w(v) \longrightarrow (wv)$ where $w = (u) \neq \lambda$ and $v$ are two Motzkin words. Conversely, let us assume that $m'$ is obtained from $m$ by a transformation $u(v) \longrightarrow (uv)$ where $u$ and $v$ are two Motzkin words. Since, a Motzkin word is obtained by the grammar $S \rightsquigarrow \lambda|(SS)$, we necessarily have $m = \alpha(u(v))\beta$ and $m' = \alpha((uv))\beta$ for some prefix and some suffix $\alpha$ and $\beta$ which directly induces that $m'$ is obtained from $m$ by a left-rotation. □

In the remainder of the paper, given $m, m' \in \mathcal{M}_n$, we write $m \longrightarrow m'$ if $m'$ is obtained from $m$ by the left-rotation defined in the previous lemma. The right-rotation will be the inverse of $\longrightarrow$. Let $\overset{*}{\longrightarrow}$ denote the reflexive and transitive closure of the rotation transformation $\longrightarrow$ in $\mathcal{M}_n$.

In order to characterize this left-rotation $\longrightarrow$, we exhibit a bijection between Motzkin words and Motzkin paths. A Motzkin path of length $n$ is a lattice path starting at $(0,0)$, ending at $(n,0)$, and never going below the $x$-axis, consisting of up steps $U = (1,1)$, horizontal steps $H = (1,0)$, and down steps $D = (1,-1)$. Let $\mathcal{P}_n$ be the set of Motzkin path of length $n-1$. It is well known that Motzkin paths are enumerated by the Motzkin numbers (A001006 in [23]).

Let $\phi$ be the bijection between $\mathcal{M}_n$ and the set $\mathcal{P}_n$ of Motzkin paths of length $n-1$ defined as follows:

- if $m = ()$ then $\phi(m) = \lambda$;
- if $m = (uv)$ where $u$, $v$ are two non-empty Motzkin words, then $\phi(m) = U\phi(v)D\phi(u)$;
- if $m = (u)$ where $u$ is a non-empty Motzkin word, then $\phi(m) = H\phi(u)$.

For instance, if $m = (()((()())))$ then $\phi(m) = UHUDD$.

**Proposition 1.** *Let $m$ and $m'$ be two Motzkin words in $\mathcal{M}_n$. Then $m \longrightarrow m'$ if and only if $\phi(m')$ is obtained from $\phi(m)$ by applying one of the two following transformations: $UH \longrightarrow HU$ and $UD \longrightarrow HH$.*

**Proof.** Let $m$ and $m'$ be two Motzkin words where $m'$ is obtained from $m$ by a left-rotation in the Tamari lattice of order $n$. By Lemma 1, we deduce that $m = \alpha(u(v))\beta$ and $m' = \alpha((uv))\beta$ where $\alpha$ and $\beta$ are some prefix and some suffix of $m$ and $m'$. Therefore $\phi(m')$ is obtained from $\phi(m)$ by replacing the factor $\phi((u(v)))$ with $\phi(((uv)))$. If $v$ is empty, then we have $\phi((u(v))) = UD\phi(u)$, $\phi(((uv))) = HH\phi(u)$ and $\phi(m')$ is obtained from $\phi(m)$ by a transformation $UD \longrightarrow HH$. If $v$ is not empty, then we have $\phi((u(v))) = UH\phi(v)D\phi(u)$, $\phi(((uv))) = HU\phi(v)D\phi(u)$ and $\phi(m')$ is obtained from $\phi(m)$ by a transformation $UH \longrightarrow HU$. This reasoning can also be considered for the converse. □