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An adjoint method for the exact calibration of stochastic local volatility models

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ABSTRACT

This paper deals with the exact calibration of semidiscretized stochastic local volatility (SLV) models to their underlying semidiscretized local volatility (LV) models. Under an SLV model, it is common to approximate the fair value of European-style options by semidiscretizing the backward Kolmogorov equation using finite differences. In the present paper we introduce an adjoint semidiscretization of the corresponding forward Kolmogorov equation. This adjoint semidiscretization is used to obtain an expression for the leverage function in the pertinent SLV model such that the approximated fair values defined by the LV and SLV models are identical for non-path-dependent European-style options. In order to employ this expression, a large non-linear system of ODEs needs to be solved. The actual numerical calibration is performed by combining ADI time stepping with an inner iteration to handle the non-linearity. Ample numerical experiments are presented that illustrate the effectiveness of the calibration procedure.

1. Introduction

In contemporary financial mathematics, *stochastic local volatility* (SLV) *models* are state-of-the-art for describing asset price processes, notably foreign exchange (FX) rates, see e.g. [21,29]. They constitute a natural combination of *local volatility* (LV) and *stochastic volatility* (SV) models. Denote by $S_\tau > 0$ the FX rate at time $\tau \geq 0$ and consider the standard transformed variable $X_\tau = \log(S_\tau/S_0)$. We deal in this paper with general SLV models of the type

$$\begin{cases} dX_\tau = (r_d - r_f - \frac{1}{2}\sigma_{SLV}^2(X_\tau, \tau)\psi^2(V_\tau))d\tau + \sigma_{SLV}(X_\tau, \tau)\psi(V_\tau)dW_\tau^{(1)}, \\ dV_\tau = \kappa(\eta - V_\tau)d\tau + \xi V_\tau^\alpha dW_\tau^{(2)}, \end{cases} \quad (1.1)$$

with $\psi(v)$ a non-negative function, α a non-negative parameter, κ, η, ξ strictly positive parameters, $dW_\tau^{(1)} \cdot dW_\tau^{(2)} = \rho d\tau$, $-1 \leq \rho \leq 1$, and given spot values S_0, V_0. The function $\sigma_{SLV}(x, \tau)$ is often called *the leverage function* and r_d, respectively r_f, denotes the risk-free interest rate in the domestic currency, respectively foreign currency. The SLV model (1.1) can be viewed as obtained from a mixture of the LV model

$$dX_{LV,\tau} = (r_d - r_f - \frac{1}{2}\sigma_{LV}^2(X_{LV,\tau}, \tau))d\tau + \sigma_{LV}(X_{LV,\tau}, \tau)dW_\tau, \quad (1.2)$$

with LV function $\sigma_{LV}(x, \tau)$, and the SV model

$$\begin{cases} dX_{SV,\tau} = (r_d - r_f - \psi^2(V_{SV,\tau}))d\tau + \psi(V_{SV,\tau})dW_\tau^{(1)}, \\ dV_{SV,\tau} = \kappa(\eta - V_{SV,\tau})d\tau + \xi V_{SV,\tau}^\alpha dW_\tau^{(2)}. \end{cases} \quad (1.3)$$

Clearly, if $\sigma_{SLV}(x, \tau)$ is identically equal to one, then the SLV model reduces to a SV model. Next, if the stochastic volatility parameter ξ is equal to zero, then the SLV model reduces to a LV model.

The choice $\psi(v) = \sqrt{v}$, $\alpha = 1/2$ corresponds to the well-known Heston-based S(L)V model, the choice $\psi(v) = v$, $\alpha = 1$ to the S(L)V model considered in [29] and the choice $\psi(v) = \exp(v)$, $\alpha = 0$ corresponds to the S(L)V model based on the exponential Ornstein–Uhlenbeck model described in [27].

If α is strictly positive, we assume that $\psi(0) = 0$ and the processes V_τ, $V_{SV,\tau}$ are non-negative. For $0 < \alpha < 1/2$ it holds that $V_\tau = 0$ is attainable, for $\alpha > 1/2$ it holds that $V_\tau = 0$ is unattainable, and for $\alpha = 1/2$ one has that $V_\tau = 0$ is attainable if $2\kappa\eta < \xi^2$, see e.g. [1]. The analogous result is true for the pure SV model (1.3).

In financial practice, $\sigma_{LV}(x, \tau)$ is determined such that the LV model (1.2) yields the exact market prices for vanilla options, see e.g. [3,6], and the parameters κ, η, ξ are chosen such that the SV model (1.3) reflects the market dynamics of the underlying asset,

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see e.g. [29]. Next, the leverage function σ_{SLV} is calibrated such that the SLV model yields the exact market prices for European call and put options. In the literature, no closed-form analytical relationship appears to be available between the leverage function and the fair value of vanilla options within the SLV model. Accordingly, in financial practice the leverage function is calibrated by making use of a relationship between the SLV model and the LV model. It is well-known, see e.g. [9,28], that these models yield the same marginal distribution for the exchange rate S_τ, and hence always define the same fair value for vanilla options, if the leverage function $\sigma_{SLV}(x, \tau)$ satisfies

$$\sigma_{LV}^2(x,\tau) = \mathbb{E}[\sigma_{SLV}^2(X_\tau,\tau)\psi^2(V_\tau)|X_\tau = x] = \sigma_{SLV}^2(x,\tau)\mathbb{E}[\psi^2(V_\tau)|X_\tau = x], \tag{1.4}$$

for all $x \in \mathbb{R}$, $\tau \geq 0$. The latter conditional expectation can be written as

$$\mathbb{E}[\psi^2(V_\tau)|X_\tau = x] = \frac{\int_{-\infty}^{\infty}\psi^2(v)p(x,v,\tau;X_0,V_0)dv}{\int_{-\infty}^{\infty}p(x,v,\tau;X_0,V_0)dv}, \tag{1.5}$$

where $p(x, v, \tau; X_0, V_0)$ denotes the joint density of (X_τ, V_τ) given by the SLV model. Since the LV function is determined such that the LV model yields exactly the observed market prices for vanilla options, the SLV model will also exactly define the same fair value whenever one is able to determine the conditional expectation above and defines the leverage function by (1.4). This conditional expectation itself depends on $\sigma_{SLV}(x, \tau)$, however, and determining it is a highly non-trivial task. Recently, a variety of numerical techniques, see e.g. [4,8,11,25,31], has been proposed in order to approximate this conditional expectation and to approximate the appropriate leverage function.

The numerical techniques presented in the references above do not take into account explicitly that, even if the LV function is known analytically, it is often not possible to determine exactly the corresponding fair value of vanilla options. Even within the LV model one relies on numerical methods in order to approximate the fair option values. A common approach consists of numerically solving the corresponding backward partial differential equation (PDE) by for example finite difference or finite volume methods, see e.g. [30]. When calibrating the SLV model to the LV model, the best result one can thus aim for is that the numerical approximation of the fair value of vanilla options is the same for both models whenever similar numerical valuation methods are used.

In this paper, we assume that the fair option value (within the LV model, resp. SLV model) is approximated through numerically solving the backward PDE (corresponding to the LV model, resp. SLV model) by standard finite difference or finite volume methods. Given such a spatial discretization for the backward PDE, an *adjoint spatial discretization* will be introduced for the corresponding forward PDE. This adjoint spatial discretization has the important property that it always defines *exactly the same approximation* for the fair value of non-path-dependent European options as the approximation given by the discretization of the backward equation. Moreover, if similar spatial discretizations are used for the backward PDE associated with the LV model and the backward PDE associated with the SLV model, then their adjoint spatial discretizations can be employed to *create an exact match between the approximations for the fair value of vanilla options within the LV model and the SLV model.*

The main contributions of this article can be visualized in the following scheme:

Discretization Backward PDE, LV	$\leftrightarrow$	Adjoint Discretization Forward PDE, LV
		$\updownarrow$ $(\star)$
Discretization Backward PDE, SLV	$\leftrightarrow$	Adjoint Discretization Forward PDE, SLV

Here relationship $(\star)$ can only be achieved if similar discretizations are used for the backward PDEs stemming from the LV and SLV models.

An outline of the rest of our paper is as follows.

In Section 2 a relationship between the forward PDE and backward PDE is introduced, both for the case of the SLV model as for the case of the LV model.

In Section 3 this relationship is preserved at the semidiscrete level: given a spatial discretization of the backward PDE, an adjoint spatial discretization for the forward PDE is defined such that both discretizations yield identical approximations for the fair value of non-path-dependent European options.

In Section 4 an actual spatial discretization, using second-order central finite difference schemes, is constructed for the backward PDE stemming from the SLV model and subsequently the corresponding adjoint spatial discretization is stated.

The main result of the paper is derived in Section 5. It is shown that, under some assumptions, the adjoint spatial discretization can be employed to obtain an expression for the leverage function such that the approximation of the fair value of vanilla options is the same for the LV and SLV models. In order to effectively use this expression, one has to solve a large system of non-linear ordinary differential equations (ODEs).

In Section 6 an Alternating Direction Implicit (ADI) temporal discretization scheme is applied to increase the computational efficiency in the numerical solution of this ODE system and in Section 7 an iteration procedure is described for handling the non-linearity.

In Section 8 ample numerical experiments are presented to illustrate the performance of the obtained SLV calibration procedure.

The final Section 9 gives concluding remarks.

2. Relationship between the forward and the backward Kolmogorov equation

Consider a European-style option with maturity T and payoff u_0. Denote by $u(x, v, t)$ the *non-discounted fair value* of the option under the SLV model (1.1) at time to maturity t, that is at time level $\tau = T - t$, if $S_\tau = S_0 \exp(x)$ and $V_\tau = v$. It is well-known, see e.g. [4], that the function u satisfies the *backward Kolmogorov equation*

$$\begin{aligned}\frac{\partial}{\partial t}u = {} & \frac{1}{2}\sigma_{SLV}^2(x,T-t)\psi^2(v)\frac{\partial^2}{\partial x^2}u + \rho\xi\sigma_{SLV}(x,T-t)\psi(v)v^\alpha\frac{\partial^2}{\partial x\partial v}u \\ & + \frac{1}{2}\xi^2v^{2\alpha}\frac{\partial^2}{\partial v^2}u + (r_d - r_f - \frac{1}{2}\sigma_{SLV}^2(x,T-t)\psi^2(v))\frac{\partial}{\partial x}u \\ & + \kappa(\eta - v)\frac{\partial}{\partial v}u,\end{aligned} \tag{2.1}$$

for $x, v \in \mathbb{R}$, $0 < t \leq T$. At maturity, i.e. at time level $\tau = T$, the initial condition $u(x, v, 0)$ is defined by the payoff u_0 of the option. By solving PDE (2.1), the fair value $e^{-r_dT}u(X_0, V_0, T)$ of the option under the SLV model can be determined at the spot, i.e. at $\tau = 0$. For strictly positive values of the parameter α, the process V_τ is non-negative and the spatial domain in the v-direction reduces to $v \geq 0$.

If the option under consideration is non-path-dependent, then the payoff u_0 is only a function of (X_T, V_T), the initial condition is given by $u(x, v, 0) = u_0(x, v)$ and the non-discounted fair value $u(x, v, t)$ of the option can be written as

$$u(x,v,t) = \mathbb{E}[u_0(X_T,V_T)|X_{T-t} = x, V_{T-t} = v],$$

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