G Model
JOCS-613; No. of Pages 14

ARTICLE IN PRESS

Journal of Computational Science xxx (2017) xxx–xxx

Contents lists available at ScienceDirect

Journal of Computational Science

journal homepage: www.elsevier.com/locate/jocs

Filling the gaps smoothly

Andrey Itkin [a,*], Alexander Lipton [b]

[a] Tandon School of Engineering, New York University, 12 Metro Tech Center, RH 517E, Brooklyn NY 11201, USA
[b] Massachusetts Institute of Technology, Connection Science, USA

ARTICLE INFO

Article history:
Received 21 November 2016
Received in revised form 6 February 2017
Accepted 8 February 2017
Available online xxx

Keywords:
Local volatility surface
Piecewise linear variance
Laplace transform
Arbitrage-free interpolation

ABSTRACT

The calibration of a local volatility models to a given set of option prices is a classical problem of mathematical finance. It was considered in multiple papers where various solutions were proposed. In this paper an extension of the approach proposed in Lipton, Sepp 2011 is developed by (i) replacing a piecewise constant local variance construction with a piecewise linear one, and (ii) allowing non-zero interest rates and dividend yields. Our approach remains analytically tractable; it combines the Laplace transform in time with an analytical solution of the resulting spatial equations in terms of Kummer's degenerate hypergeometric functions.

The local volatility model introduced by [8,7] is a classical model of mathematical finance. The calibration of the local volatility (LV) surface to the market data, representing either prices of European options or the corresponding implied volatilities for a given set of strikes and maturities, drew a lot of attention over the past two decades. Various approaches to solving this important problem were proposed, see, e.g., [3,19,13] and references therein. Below, we refer to [19] as LS2011 for the sake of brevity.[1]

There are two main approaches to solving the calibration problem. The first approach attempts to construct a continuous implied volatility (IV) surface matching the market quotes by using either some parametric or non-parametric regression, and then generates the corresponding LV surface via the well-known Dupire formula, see, e.g., [13] and references therein. To be practically useful, this construction should guarantee no arbitrage for all strikes and maturities, which is a serious challenge for any model based on interpolation. If the no-arbitrage condition is satisfied, then the LV surface can be calculated using (2) below, which is equivalent to, but more convenient than, the original Dupire formula. The second approach relies on the direct solution of the Dupire equation using either analytical or numerical methods. The advantage of this approach is that it guarantees no-arbitrage. However, the problem of the direct solution can be ill-posed, [4], and is rather computationally expensive.

An additional difficulty with both approaches is that the calibration algorithm has to be fast in order to be practically useful. On the one hand, analytical or numerical solutions of the Dupire equation are naturally numerically expensive. On the other hand, building a no-arbitrage IV surface could also be surprisingly numerically challenging, because it requires solving a rather involved constrained optimization problem, see [13]. An additional complication arises from the fact that in the wings the implied variance surface should be at most linear in the normalized strike [15].

In this paper we extend the approach proposed in LS2011, which is based on the direct solution of the transformed Dupire equation. In LS2011 a piecewise constant LV surface is chosen, and an efficient semi-analytical method for calibrating this surface to the sparse market data is proposed. However, one can argue that ideally the LV function should be continuous in the log-strike space. Below we demonstrate how to extend LS2011 approach by assuming that the local variance is piecewise linear in the log-strike space, so that the corresponding LV surface is continuous in the strike direction (but not in the time direction). While derivatives of the LV function with respect to strike have discontinuities, the option prices, deltas and gammas are continuous. This is to compare with LS2011 where the option prices and deltas are continuous while the option gammas are discontinuous. We also allow for non-zero interest rates and proportional dividends.

The rest of the paper is organized as follows. Section 1 introduces the Dupire equation and discusses a general approach to constructing the LV surface. Section 2 considers all necessary steps for solving the Dupire equation. Section 3 introduces a no-arbitrage interpolation of the source term, which naturally appears when the Laplace

* Corresponding author.
E-mail address: itkin@chem.ucla.edu (A. Itkin).

[1] We emphasize that the solution proposed in [3] is static in nature, while the solution developed in LS2011 is fully dynamic.

http://dx.doi.org/10.1016/j.jocs.2017.02.003

transform in time is used, and shows that using this interpolation all the integrals containing this source term can be obtained in a closed form. Section 4 considers a special case when the slope of the local variance on some interval is small, so the linear local variance function on this interval becomes flat. Section 5 discusses various asymptotic results which are useful for constructing the general solution of the Dupire equation. Section 6 is devoted to the calibration of the model and also describes how to get an educated initial guess for the optimizer. Since computing the inverse Laplace transform could be expensive for small time intervals, Section 7 describes an asymptotic solution obtained in this limit in [11] and shows how to use it for our purposes. Section 8 describes numerical results for a particular set of market data. The final section concludes. Some additional proofs and derivations are given in two appendices.

1. Local volatility surface

As a general building block for constructing the local volatility surface we consider Dupire's (forward) equation for the Put option price P which is a function of the strike price K and the time to maturity T [8]. We assume that the underlying stock process S_t under the risk neutral measure is governed by the following stochastic differential equation

$$dS_t = (r-q)S_t dt + \sigma(S_t, t)S_t dW_t, \qquad S_0 = S,$$

where $r \geq 0$ is a constant risk free rate, $q \geq 0$ is a constant continuous dividend yield, σ is a given local volatility function, and W_t is the standard Brownian motion. The Dupire equation for the Put $P(K, T)$ reads [9]

$$P_T(K,T) = \left\{ \frac{1}{2}\sigma^2(K,T)K^2 \frac{\partial^2}{\partial K^2} - (r-q)K\frac{\partial}{K} - q \right\} P(K,T), \tag{1}$$

$$(K, T) \in (0, \infty) \times [0, \infty),$$

subject to the initial and boundary conditions

$$P(K, 0) = (K - S_0)^+,$$

$$P(0, T) = 0, \quad P(K, T)_{K\uparrow\infty} = KD, \quad D = e^{-rT},$$

where $S_0 = S_t|_{t=0}$, and D is the discount factor.

If the market quotes for $P(K, T)$ are known for all K, T, then the LV function $\sigma(K, T)$ can be uniquely determined everywhere by inverting (1).[2] However, in practice, the known set of market quotes is a discrete set of pairs (K_i, T_j), $i = 1, \ldots, n_j$, $j = 1, \ldots, M$, where n_j is the total number of known quotes for the maturity T_j, which obviously does not cover all K, T. So the form of $\sigma(K, T)$ remains unknown.

In order to address this issue, it is customary to choose a functional form of $\sigma(K, T)$ for the corresponding time slice. For instance, in LS2011 $\sigma(K, T)$ is assumed to be a piecewise constant function of K, T. The authors propose a general methodology of solving (1) for their chosen explicit form of $\sigma(K, T)$ by using the Carson–Laplace transform in time and Green's function method in space. This opens the door for using a version of the least-square method for the calibration routine. Of course, by construction, it makes the whole local volatility surface discontinuous at the boundaries of the tiles, and flat in the wings. While the former feature, in itself, is not necessarily an issue, but should be avoided if possible, the latter feature is somewhat more troubling, since, it is shown in [6,12], that the asymptotic behavior of the local variance is linear in the log strike at both $K \to \infty$ and $K \to 0$. While the result for $K \to 0$ is shown to be true at least for the Heston and Stein–Stein models, the result for $K \to \infty$ directly follows from Lee's moment formula for the implied variance v_I, [15], and the representation of σ^2 via the total implied variance $w = v_I T$ [17,10]

$$w_L \equiv \sigma^2(T, K)T = \frac{T\partial_T w}{\left(1 - \frac{X\partial_X w}{2w}\right)^2 - \frac{(\partial_X w)^2}{4}\left(\frac{1}{w} + \frac{1}{4}\right) + \frac{\partial_X^2 w}{2}}, \tag{2}$$

where $w = w(X, T)$, $X = \log K/F$ and $F = Se^{(r-q)T}$ is the stock forward price. Therefore, having a flat local volatility deep in the wings should be avoided if possible.

That is why, in this paper, we consider a continuous, piecewise linear local variance $v = \sigma^2(X, T)$ in the spatial variable X for a fixed $T = const$. This allows us to match the asymptotic behavior of v in the wings as well as build the whole surface which is much smoother than in the piecewise constant case. Also, in LS2011 the interest rates and dividends are assumed to be zero, while here we take them into account.

2. Solution of Dupire's equation

Introducing a new dependent variable

$$B(X, T) = e^{-X/2}(KD - P(X, T))/Q, \qquad Q = Se^{-qT},$$

which is a scaled covered Put, the problem in (1) can be re-written as follows

$$\begin{aligned} & B_T - \frac{1}{2}vB_{XX} + \frac{1}{8}vB = 0, \quad B(X, 0) = \frac{K - (K-S)^+}{S}e^{-X/2} \\ & = e^{-X/2}\mathbf{1}_{X>0} + e^{X/2}\mathbf{1}_{X\leq 0}, \quad B(X,T)_{X\downarrow-\infty} = 0, \quad B(X,T)_{X\uparrow\infty} = 0, \\ & (X, T) \in (-\infty, \infty) \times [0, \infty). \end{aligned} \tag{3}$$

A similar transformation is used in [18] in order to solve the backward Black–Scholes equation. Suppose that there are option price quotes (at least for one strike) for M different maturities $T_1, \ldots, T_M$.[3] Also suppose that for each T_j the market quotes are provided at X_i, $i = 1, \ldots, n_j$.[4] Then the corresponding continuous piecewise linear local variance function $v_j(X)$[5] on the interval $[X_i, X_{i+1}]$ reads

$$v_{j,i}(X) = v^0_{j,i} + v^1_{j,i}X, \tag{4}$$

where we use the super-index 0 to denote a level v^0, and the super-index 1 to denote a slope v^1. Subindex $i = 0$ in $v^0_{j,0}$, $v^1_{j,0}$ corresponds to the interval $(-\infty, X_1]$. Since $v_j(X)$ is continuous, we have

$$v^0_{j,i} + v^1_{j,i}X_{i+1} = v^0_{j,i+1} + v^1_{j,i+1}X_{i+1}, \quad i = 0, \ldots, n_j - 1. \tag{5}$$

The first derivative of $v_j(X)$ experiences a jump at the points X_i, $i \in \mathbb{Z} \cap [1, n_j]$.

Further, assume that $v(X, T)$ is a piecewise constant function of time, i.e. $v^0_{j,i}$, $v^1_{j,i}$ do not depend on T on the intervals $[T_j, T_{j+1})$, $j \in [0, M-1]$, and jump to new values at the points T_j, $j \in \mathbb{Z} \cap [1, M]$. In the original independent variables K, T this condition implies that

$$v(K_i, T) \equiv v_{j,i} = v^0_{j,i} + v^1_{j,i}\left[\log(K_i/S) - (r-q)T\right], \quad T \in [T_j, T_{j+1}),$$

i.e. that the local variance is a (discontinuous) piecewise linear function of time T. In other words, in the original log-variables $(\log K, T)$ the function $v(\log K, T)$ is piecewise linear in both variables, while

[2] If the Call option market prices are given for some strikes and maturities, we can use Call-Put parity in order to convert them to Put prices, since for calibration we usually use vanilla European option prices.

[3] We assume the maturities are sorted in the increasing order.

[4] The strikes also are assumed to be sorted in the increasing order.

[5] Here in the notation we drop off the dependence of v on T since T is given, and hopefully it does not bring any confusion.