Journal of Computational Science 11 (2015) 123–136

Contents lists available at ScienceDirect

Journal of Computational Science

journal homepage: www.elsevier.com/locate/jocs

Performance evaluation of block-diagonal preconditioners for the divergence-conforming B-spline discretization of the Stokes system

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ARTICLE INFO

Article history:
Received 2 November 2014
Received in revised form
23 December 2014
Accepted 28 January 2015
Available online 20 February 2015

Keywords:
Isogeometric analysis
Divergence-conforming B-spline spaces
Krylov subspace method
Block-diagonal preconditioner
Stokes problem

ABSTRACT

The recently introduced divergence-conforming B-spline discretizations allow the construction of smooth discrete velocity–pressure pairs for viscous incompressible flows that are at the same time inf-sup stable and pointwise divergence-free. When applied to discretized Stokes equations, these spaces generate a symmetric and indefinite saddle-point linear system. Krylov subspace methods are usually the most efficient procedures to solve such systems. One of such methods, for symmetric systems, is the Minimum Residual Method (MINRES). However, the efficiency and robustness of Krylov subspace methods is closely tied to appropriate preconditioning strategies. For the discrete Stokes system, in particular, block-diagonal strategies provide efficient preconditioners. In this article, we compare the performance of block-diagonal preconditioners for several block choices. We verify how the eigenvalue clustering promoted by the preconditioning strategies affects MINRES convergence. We also compare the number of iterations and wall-clock timings. We conclude that among the building blocks we tested, the strategy with relaxed inner conjugate gradients preconditioned with incomplete Cholesky provided the best results.

1. Introduction

The concept of isogeometric analysis (IGA) first appeared in [1], and since then several papers followed, either exploring their mathematical theory, for example [2,3], or showing their potential in engineering applications, to mention some [4–12]. In [13], the IGA concept is used to discretize vector fields of electromagnetic problems. For such problems, it is known that the function spaces satisfy a de Rham diagram at the continuous level, and for a discretization to be successfully applied to them, the finite dimensional spaces should also satisfy the de Rham diagram at the discrete level. Exploring one of the main features of spline basis functions, that is the easy control of the basis polynomial degree and regularity, and by a suitable choice of B-spline spaces of each component of the two-dimensional vector field, Buffa et al. [13] introduced an IGA discretization satisfying a de Rham diagram. They have shown that the technique can be viewed as a smooth generalization of Nédélec elements, and thus good results were reported.

The generalization for three-dimensional vector fields and the mathematical theory of such discretization appeared in [14]. Their approach, called Isogeometric Discrete Differential Forms, was inspired by the theory of finite element exterior calculus of Arnold et al. [15].

In [16], Buffa et al. introduced three similar vector field discretizations for the Stokes problem. By a proper choice of the polynomial degrees and the regularity of the components of the discrete velocity field and the discrete pressure field, these discretizations can be interpreted as smooth generalizations of Nédélec, Taylor-Hood and Raviart-Thomas elements. Because of the smoothness of the basis functions used, the discrete velocity spaces of these elements are $\mathbf{H}^1$-conforming, which make them suitable to discretize the Stokes system. Furthermore, in the case of the Raviart-Thomas element type, Buffa et al. [16] characterize the image of the divergence operator from the discrete velocity space (with and without boundary conditions) onto the discrete pressure space, guaranteeing this way a point-wise divergence-free discrete vector field, a condition

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http://dx.doi.org/10.1016/j.jocs.2015.01.005
1877-7503/

that is generally only satisfied weakly by classical mixed finite elements.

Following the developments of Buffa et al. [14], Evans and Hughes [17] further developed the Raviart-Thomas element type in the context of Hilbert complexes. Indeed, by using the stable projectors of [14], a divergence-preserving transformation (Piola transformation) of the velocity field and an integral-preserving transformation of the pressure field, Evans and Hughes devised a Stokes complex with a compatible sub-complex that furnishes a discretization scheme, that is at the same time *inf-sup* stable and divergence free. In [17–19], Evans and Hughes applied this discretization scheme to several viscous incompressible flows, and also started its mathematical theory as well.

The discretization of the Stokes equations by *inf-sup* stable mixed elements requires the solution of a symmetric indefinite linear system, called the (discrete) Stokes system, with a block coefficient matrix of saddle-point type. Several strategies for solving the Stokes linear system have appeared in the literature [20–23], the most popular being variants of Uzawa's method, such as the *inexact* Uzawa method, and the Minimum Residual Method (MINRES) [24]. The latter is a member from the family of Krylov subspace methods, and as such, its robustness and performance is very dependent on the preconditioning strategy.

For example, MINRES is being used to solve large-scale problems in science, such as Earth's mantle convection flows in parallel by finite elements with octree-based adaptive mesh refinement and coarsening (AMR/C), demonstrating scalability up to 122,880 cores [25].

The rest of the article is organized as follows. In Section 2, we review some isogeometric analysis definitions, in order to setup the nomenclature for the divergence-conforming discretization. Section 3 reviews the results of [17] with respect to Stokes flow. First, we present the discrete velocity–pressure pair on the parametric domain, and how it is mapped to general geometries by means of proper transformations. Also, the inf-sup stability and the divergence-free property of the divergence-conforming discrete velocity–pressure pair is presented. The next section deals with the discrete variational problem, and how Nitsche's method is used to impose Dirichlet boundary conditions weakly. In Section 5, we review the Minimum Residual Method. We discuss its convergence properties and how to precondition it. We also present the block-diagonal preconditioning strategy introduced by Wathen and Silvester in [26,27], and the choices we made for the preconditioners blocks. Section 6 describes our numerical results. We present the results for three examples: two manufactured analytical solutions for different geometries and the lid-driven cavity flow benchmark. For the lid-driven problem we analyze the preconditioners performance.

2. Isogeometric notation: spline spaces and the geometrical mapping

We recall some spline spaces definition and related notation to describe the divergence-conforming spaces introduced in [17]. Here, we follow closely [16,14,17].

2.1. Univariate B-splines

To define a univariate B-spline basis we specify the number n of basis functions wanted, the polynomial degree p of the basis and a knot vector Ξ. A knot vector Ξ is a finite nondecreasing sequence $\Xi = \{0 = \xi_1, \ldots, \xi_{n+p+1} = 1\}$. The sequence may have repeated knots, in this case one says that the knot has multiplicity greater than one. Introducing the vector $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_m\}$ of knots without repetitions, also called breakpoints, and the vector $\{r_1, \ldots, r_m\}$ of their corresponding multiplicities, one has that,

$$\Xi = \{\underbrace{\zeta_1, \ldots, \zeta_1}_{r_1 \text{ times}}, \underbrace{\zeta_2, \ldots, \zeta_2}_{r_2 \text{ times}}, \ldots, \underbrace{\zeta_m, \ldots, \zeta_m}_{r_m \text{ times}}\}, \tag{1}$$

with $\sum_{i=1}^{m} r_i = n + p + 1$.

The B-spline basis functions are p-degree piecewise polynomials on the subdivision $\{\zeta_1, \ldots, \zeta_m\}$. A stable way of generating them is by using the Cox-de Boor recursion algorithm [28], which receives as inputs p and Ξ. Knot multiplicity is an essential ingredient in spline theory, which controls the basis smoothness. Indeed, if a breakpoint ζ_j has multiplicity r_j, then the basis functions have at least $\alpha_j := p - r_j$ continuous derivatives at ζ_j. Hence, the maximum multiplicity allowed for ζ_j is $r_j = p + 1$, in this case $\alpha_j = -1$ and the basis is discontinuous at ζ_j. We restrict ourselves to open knot vectors, in this case $r_1 = r_m = p + 1$, which implies $n \geq p + 1$ and $\alpha_1 = \alpha_m = -1$. The vector $\boldsymbol{\alpha} := \{\alpha_1, \ldots, \alpha_m\}$ collects the basis regularity. We define $\boldsymbol{\alpha} - 1 = \{-1, \alpha_2 - 1, \ldots, \alpha_{m-1} - 1, -1\}$, when $\alpha_j \geq 0$ for $2 \leq j \leq m - 1$, and $|\boldsymbol{\alpha}| = \min\{\alpha_2, \ldots, \alpha_{m-1}\}$.

The set $\left\{B_i^p\right\}_{i=1}^{n}$ defines a linearly independent set of functions with all the good properties wanted for analysis purposes [29]. The space of B-splines spanned by them is denoted by,

$$\mathcal{S}_{\boldsymbol{\alpha}}^{p} := \operatorname{span}\left\{B_i^p\right\}_{i=1}^{n}. \tag{2}$$

For univariate spline spaces, when $p \geq 1$ and $\alpha_j \geq 0$ for $2 \leq j \leq m - 1$, the derivative of a spline is a spline too, indeed the derivative is a surjective operator, that is,

$$\left\{\frac{d}{dx}u : u \in \mathcal{S}_{\boldsymbol{\alpha}}^{p}\right\} \equiv \mathcal{S}_{\boldsymbol{\alpha}-\mathbf{1}}^{p-1}. \tag{3}$$

2.2. Bivariate B-splines

Given p_1, p_2, n_1, n_2, and the knot vectors Ξ_1 and Ξ_2, we construct a univariate B-spline basis in each direction, that is, $\{B_{i_d,d}^{p_d}\}_{i_d=1}^{n_d}$ for $d = 1, 2$. The bivariate B-spline basis functions are defined by tensor products of the univariate ones as

$$B_{i_1,i_2}^{p_1,p_2} := B_{i_1,1}^{p_1} \otimes B_{i_2,2}^{p_2}, \quad i_1 = 1, \ldots, n_1; i_2 = 1, \ldots, n_2. \tag{4}$$

The breakpoints $\boldsymbol{\zeta}_d = \{\zeta_{1,d}, \ldots, \zeta_{m_d,d}\}$ in each direction $d = 1, 2$ define a mesh

$$\begin{aligned}\mathcal{M}_h = \{Q = (\zeta_{i_1,1}, \zeta_{i_1+1,1}) \\ \times (\zeta_{i_2,2}, \zeta_{i_2+1,2}) : 1 \leq i_1 \leq m_1 - 1, 1 \leq i_2 \leq m_2 - 1\},\end{aligned} \tag{5}$$

called the parametric mesh, on the parametric domain $\widehat{\Omega} = (0, 1)^2$. The subscript h stands for the global mesh size, and is defined as $h := \max_{Q \in \mathcal{M}_h} h_Q$, where $h_Q := \operatorname{diam}(Q)$. To guarantee theoretical convergence estimates, the mesh $\mathcal{M}_h$ should satisfy a shape-regularity condition [2],

$$\lambda^{-1} \leq \frac{h_{Q,\min}}{h_Q} \leq \lambda, \qquad \forall Q \in \mathcal{M}_h, \tag{6}$$

for constant $\lambda > 0$, where $h_{Q,\min}$ is the length of the smallest edge of Q. If the same λ holds for a sequence of nested refined meshes $\{\mathcal{M}_h\}_{h \leq h_0}$, this sequence is said to be locally quasi-uniform, which we assume hereafter.

Using the notation $\boldsymbol{\alpha}_1 = \{\alpha_{1,1}, \ldots, \alpha_{m_1,1}\}$ and $\boldsymbol{\alpha}_2 = \{\alpha_{1,2}, \ldots, \alpha_{m_2,2}\}$ for the regularity vectors in each direction, the bivariate B-spline space is defined as

$$\mathcal{S}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{p_1,p_2} \equiv \mathcal{S}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{p_1,p_2}(\mathcal{M}_h) := \operatorname{span}\left\{B_{i_1,i_2}^{p_1,p_2}\right\}_{i_1,i_2=1}^{n_1,n_2}. \tag{7}$$

The global regularity of the space is defined as $\alpha := \min\{|\boldsymbol{\alpha}_1|, |\boldsymbol{\alpha}_2|\}$.