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A single-exponential fixed-parameter algorithm for distance-hereditary vertex deletion ☆

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Vertex deletion problems ask whether it is possible to delete at most k vertices from a graph so that the resulting graph belongs to a specified graph class. Over the past years, the parameterized complexity of vertex deletion to a plethora of graph classes has been systematically researched. Here we present the first single-exponential fixed-parameter tractable algorithm for vertex deletion to distance-hereditary graphs, a well-studied graph class which is particularly important in the context of vertex deletion due to its connection to the graph parameter rank-width. We complement our result with matching asymptotic lower bounds based on the exponential time hypothesis. As an application of our algorithm, we show that a vertex deletion set to distance-hereditary graphs can be used as a parameter which allows single-exponential fixed-parameter tractable algorithms for classical NP-hard problems.

1. Introduction

Vertex deletion problems include some of the best studied NP-hard problems in theoretical computer science, including Vertex Cover or Feedback Vertex Set. In general, the problem asks whether it is possible to delete at most k vertices from a graph so that the resulting graph belongs to a specified graph class. While these problems are studied in a variety of contexts, they are of special importance for the parameterized complexity paradigm [13,11], which measures the performance of algorithms not only with respect to the input size but also with respect to an additional numerical parameter. The notion of vertex deletion allows a highly natural choice of the parameter (specifically, k), especially for problems where the solution size is not defined or cannot be used. Many vertex deletion problems are known to admit so-called *single-exponential fixed-parameter tractable (FPT) algorithms*, which are algorithms running in time $\mathcal{O}(c^k \cdot n^{\mathcal{O}(1)})$ for input size n and some constant c.

Over the past years, the parameterized complexity of vertex deletion to a plethora of graph classes has been systematically researched, and in particular, if the target class admits efficient algorithms for many NP-hard problems, then such a class get more attention. For this reason, classes of graphs of constant treewidth have been studied in detail, and Fomin et al. [17] and Kim et al. [34] showed that the corresponding Treewidth-t Vertex Deletion[1] problem is solvable

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[1] Treewidth-t Vertex Deletion asks whether it is possible to delete k vertices so that the resulting graph has treewidth at most t.

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in single-exponential FPT time. Interestingly, this problem is a special case of general Planar $\mathcal{F}$-Deletion problems, which ask whether one can hit all of minor models of graphs in $\mathcal{F}$ by at most k vertices, when $\mathcal{F}$ contains at least one planar graph. The condition that $\mathcal{F}$ contains a planar graph is essential because it tells that the outside of any solution should have bounded treewidth, by the grid-minor theorem [39]. Several authors [17,34] have used this fact to design single-exponential FPT algorithms.

The successful development of single-exponential FPT algorithms for Treewidth-t Vertex Deletion motivates us to study Rank-width t-Vertex Deletion, which is analogous to Treewidth-t Vertex Deletion but replaces treewidth with rank-width. Rank-width [37,36] is a graph parameter introduced for generalizing graph classes of bounded treewidth into dense graph classes; for example, complete graphs have unbounded treewidth but rank-width 1. Generally, classes of graphs of bounded rank-width capture the graphs that can be recursively decomposable along vertex bipartitions (A, B) where the number of distinct neighborhood types from one side to the other is bounded. Courcelle et al. [8] proved that every MSO_1-expressible problem can be solved in polynomial time on graphs of bounded rank-width (see also the work of Ganian and Hliněný [20]).

Kanté et al. [31] observed that Rank-width t-Vertex Deletion is fixed parameter tractable using the general framework of Courcelle, Makowski, and Rotics. However, this algorithm does not provide any reasonable function for k. Thus Kanté et al. naturally asked whether it is solvable in reasonably better running time. For instance, it is actually open whether Rank-width t-Vertex Deletion can even be solved in time $2^{2^{\mathcal{O}(k)}}n^{\mathcal{O}(1)}$, where k is the size of the deletion set.

In this paper, we focus on graphs of rank-width at most 1, which are *distance-hereditary graphs*. Distance-hereditary graphs were introduced by Howorka [25] in 1977, long before the discovery of rank-width [37] and the observation by Oum [36] that the class of graphs of rank-width at most 1 are precisely distance-hereditary graphs. Bandelt and Mulder [3] found all the minimal induced subgraph obstructions for distance-hereditary graphs. Distance-hereditary graphs are naturally related to split decompositions, where they are exactly the graphs that are completely decomposable into stars and complete graphs [4]. We explain these structural properties in more detail in Section 2. This structure has led to the development of a number of algorithms for distance-hereditary graphs [7,28,26,29,40,35,21]. Given the above, we view the vertex deletion problem for distance-hereditary graphs as a first step towards handling Rank-width-t Vertex Deletion.

Our Contribution.

A graph G is called *distance-hereditary* if for every connected induced subgraph H of G and every $v, w \in V(H)$, the distance between v and w in H is the same as the distance between v and w in G. We study the following problem.

Distance-Hereditary Vertex Deletion *Instance:* A graph G and an integer k. *Parameter:* k. *Task:* Is there a vertex set $Q \subseteq V(G)$ with $\lvert Q\rvert \le k$ such that $G - Q$ is distance-hereditary?

The main result of this paper is a single-exponential FPT algorithm for Distance-Hereditary Vertex Deletion.

Theorem 1.1. Distance-Hereditary Vertex Deletion *can be solved in time* $\mathcal{O}(37^k \cdot |V(G)|^7(|V(G)| + |E(G)|))$.

We note that this solves an open problem of Kanté et al. [31]. The core of our approach exploits two distinct characterizations of distance-hereditary graphs: one by forbidden induced subgraphs (obstructions), and the other by admitting a special kind of split decomposition [9].

The algorithm can be conceptually divided into three parts.

1. **Iterative Compression**. This technique allows us to reduce the problem to the easier Disjoint Distance-Hereditary Vertex Deletion, where we assume that the instance additionally contains a certain form of advice to aid us in our computation. Specifically, this advice is a vertex deletion set S to distance-hereditary graphs which is disjoint from and slightly larger than the desired solution.
2. **Branching Rules**. We exhaustively apply two branching rules to simplify the given instance of Disjoint Distance-Hereditary Vertex Deletion. At a high level, these branching rules allow us to assume that the resulting instance contains no small obstructions and furthermore that certain connectivity conditions hold on $G[S]$.
3. **Simplification of Split Decomposition**. We compute the split decomposition of $G - S$ and exploit the properties of our instance G guaranteed by branching to prune the decomposition. In particular, we show that the connectivity conditions and non-existence of small obstructions mean that S must interact with the split decomposition of $G - S$ in a special way, and this allows us to identify irrelevant vertices in $G - S$. This is by far the most technically challenging part of the algorithm.

A more detailed explanation of our algorithm is provided in Section 3, after the definition of required notions. We complement this result with an algorithmic lower bound which rules out a subexponential FPT algorithm for Distance-Hereditary Vertex Deletion under well-established complexity assumptions. We also note that the naive approach of simply hitting all known “obstructions” (i.e., forbidden induced subgraphs) for distance-hereditary graphs does not lead to an FPT