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The tight asymptotic approximation ratio of First Fit for bin packing with cardinality constraints

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ABSTRACT

In bin packing with cardinality constraints (BPCC), there is an upper bound $k \geq 2$ on the number of items that can be packed into each bin, additionally to the standard constraint on the total size of items. We study the algorithm First Fit (FF), acting on a list of items, packing each item into the minimum indexed bin that contains at most $k-1$ items and has sufficient space for the item. We present a complete analysis of its asymptotic approximation ratio for all values of k. Many years after FF for BPCC was introduced, its tight asymptotic approximation ratio is finally found.

1. Introduction

Bin packing with cardinality constraints (BPCC) is a variant of standard bin packing [25,13–15,19,28,21]. The input consists of items, denoted by $1, 2, \ldots, n$, where item i has a size $\sigma_i > 0$ associated with it, and a parameter $k \geq 2$, called the cardinality constraint. The value k is seen as a constant while n is a variable. The goal is to partition the input items into subsets called bins, such that the total size of items of one bin does not exceed 1, and the number of items does not exceed k. In many applications of bin packing, the assumption that a bin can contain any number of items is not realistic, and bounding the number of items as well as their total size provides a more accurate modeling of the problem. BPCC was studied both in the offline and online environments [17,18,16,6,1,8,10,11,5].

In this paper we study the algorithm First Fit (FF). This algorithm processes the input items one by one. Each item is packed into the a bin of the smallest index where it can be packed. An item i can be packed into bin B if the packing is possible both with respect to the total size of items already packed into that bin and with respect to the number of packed items, i.e., the bin contains items of total size at most $1 - \sigma_i$ and it contains at most $k-1$ items. An optimal algorithm that packs the items into a minimum number of bins is denoted by OPT. For an input L and algorithm A, we let $A(L)$ denote the number of bins that A uses to pack L. We also use $OPT(L)$ to denote the number of bins that OPT uses for a given input L. The algorithm A can be offline (receiving the items as a set) or online (the items are presented one at a time, and the next item is not presented until the item presented just before it is packed), while OPT is offline. FF can be seen as an offline algorithm or as an online algorithm. The absolute approximation ratio of an algorithm A is the supremum ratio over all inputs L between the number of bins $A(L)$ (that it uses) and the number of bins $OPT(L)$ (that OPT uses). The asymptotic approximation ratio is the limit of absolute approximation ratios R_K when K tends to infinity and R_K takes into

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Table 1
Bounds for $2 \le k \le 12$. The second column contains the tight asymptotic approximation ratio of FF, the third column contains the previous upper bound on FF's asymptotic approximation ratio, the fourth column contains the tight asymptotic approximation ratio of Harmonic, and the last column contains the asymptotic approximation ratio of the current best algorithm. Entries without a citation are those proved here.

Value of k	FF (some lower bounds are folklore)	prev. UB for FF	Harmonic	Best known UB
2	1.5	1.5 [17]	1.5 [8]	1.44721 [1]
3	1.8333	1.8333 [5]	1.8333 [8]	1.75 [8]
4	2	2 [5]	2 [8]	1.86842 [8]
5	2.1333	2.22 [17]	2.1 [8]	1.93719 [8]
6	2.2222	2.3 [17]	2.16667 [8]	1.99306 [8]
7	2.2857	2.35714 [17]	2.2381 [8]	2 [1]
8	2.3333	2.4 [17]	2.29167 [8]	2 [1]
9	2.3704	2.43333 [17]	2.33333 [8]	2 [1]
10	2.4	2.46 [17]	2.36666 [8]	2 [1]
11	2.4273	2.481818 [17]	2.39394 [8]	2 [1]
12	2.45	2.5 [17]	2.41667 [8]	2 [1]

account only inputs for which OPT uses at least K bins. The term *competitive ratio* is sometimes used for online algorithms instead of *approximation ratio* and it is equivalent. We use the term approximation ratio throughout the paper, and when we do not write whether we mean the asymptotic approximation ratio or the absolute approximation ratio, we mean the former. We see a bin as a set of items, and for a bin B, we let $\sigma(B) = \sum_{i \in B} \sigma_i$ be its level.

Bin packing problems are often studied with respect to asymptotic measures. Approximation algorithms were designed for the offline version of BPCC (which is strongly NP-hard for $k \ge 3$) [17,16,6,10], and the problem has an asymptotic fully polynomial approximation scheme (AFPTAS) [6,10]. Using elementary bounds, it was shown by Krause, Shen, and Schwetman [17] that FF has an asymptotic approximation ratio of at most $2.7 - \frac{2.4}{k}$ (they also presented a lower bound on the asymptotic approximation ratio, which is not tight, but tends to 2.7 as k grows to infinity). For $k \to \infty$, it can be deduced that the asymptotic approximation ratio is 2.7 also since this is a special case of vector bin packing (with at least two dimensions) [12], where the first components of (vector) items are equal to the item sizes (of items of an input for BPCC), and the second components are equal to $\frac{1}{k}$ (the remaining components are equal to zero or to $\frac{1}{k}$). The case $k = 2$ is solvable using FFD (the version of FF where items are first sorted by non-increasing size) and by other methods in the offline scenario, but it is not completely resolved in the online scenario, and the best possible asymptotic approximation ratio is in $[1.42764, 1.44721]$ [20,1,11]. For larger k, there is an approximation algorithm of absolute approximation ratio at most 2 [1,5], and improved algorithms (that have smaller asymptotic approximation ratios than $\min\{2, 2.7 - \frac{2.4}{k}\}$) are known for $k = 3, 4, 5, 6$ [8]. A full analysis of the cardinality constrained variant of the Harmonic algorithm [19] (that partitions items into k classes and packs each class independently are separately (greedily), such that the classes are $I_\ell = (\frac{1}{\ell+1}, \frac{1}{\ell}]$ for $1 \le \ell \le k-1$ and $I_k = (0, \frac{1}{k}]$, and for any $1 \le \ell \le k$, each bin of I_ℓ, possibly except for the last such bin, receives exactly ℓ items) is given in [8], and its approximation ratio for $k = 2, 3$ is 1.5 and $\frac{11}{6}$, respectively (its approximation ratio is in $[2, 2.69103]$ for $k \ge 4$, see Table 1 for some additional values). Known lower bounds on the approximation ratio did not exceed those known for standard bin packing [28,26,4,11,5] until recently, but now it is known that the overall best possible bound for the absolute and asymptotic approximation ratios is 2 [1,5,2] (and improved lower bounds for specific values of k were proved as well [2]).

For standard online bin packing, it is known that the best asymptotic approximation ratio is in [1.5403, 1.57829] [26,4,22,3], the asymptotic (and absolute) approximation ratio of FF is 1.7 [15,7]. Another related problem is called *class constrained bin packing* [9,23,24,27]. In that problem each item has a color, and a bin cannot contain items of more than k different colors (for a fixed parameter k). BPCC is the special case of that problem where all items have distinct colors.

In this paper we provide a complete analysis of the famous and natural algorithm FF with respect to the asymptotic approximation ratio. We find that the asymptotic approximation ratio of FF is $2.5 - \frac{2}{k}$ for $k = 3, 4$, $\frac{8(k-1)}{3k} = \frac{8}{3} - \frac{8}{3k}$ for $4 \le k \le 10$, and $2.7 - \frac{3}{k}$ for $k \ge 10$ (recall that the values $k = 4$ and $k = 10$ are included in two cases each). Interestingly, introducing cardinality constraints (with sufficiently large values of k) results in an increase of some approximation ratios by 1 [17,15,19,8]. In particular, both the approximation ratio of the cardinality constrained Harmonic algorithm and FF have approximation ratios that are larger by 1 than their approximation ratios for standard bin packing. Thus, Harmonic has a slightly smaller approximation ratio of 2.69103. Moreover, it can be verified that the worst-case examples of Harmonic are valid (but not tight) for FF. For $k = 2, 3, 4$ they have the same approximation ratios, but not for $k \ge 5$, and in many cases the approximation ratio of Harmonic is much smaller (see examples in Table 1). Previous results on the absolute approximation ratio of FF [5] imply the upper bounds which we prove for the cases $k = 3, 4$, and the upper bound for the case $k = 2$ was known [17]. We provide these results for completeness, and as an introduction to the other cases. The proofs of these cases are much simpler than those given previously [5], since most of the focus in that proof was to deal with additive constants.

While FF is a frequently studied natural algorithm, its exact asymptotic approximation ratio as a function of k was unknown. While it is not difficult to show an upper bound of 2.7 for all values of k [17,12], providing such a tight analysis as a function of k turns out to be quite difficult. Intuitively, it initially seems that the asymptotic approximation ratio should simply increase by $\frac{k-3}{k}$ compared to the approximation ratio of FF for standard bin packing. The reason for this is that in