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k-distinct in- and out-branchings in digraphs ☆

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ABSTRACT

An out-branching and an in-branching of a digraph D are called k-distinct if each of them has k arcs absent in the other. Bang-Jensen, Saurabh and Simonsen (2016) proved that the problem of deciding whether a strongly connected digraph D has k-distinct out-branching and in-branching is fixed-parameter tractable (FPT) when parameterized by k. They asked whether the problem remains FPT when extended to arbitrary digraphs. Bang-Jensen and Yeo (2008) asked whether the same problem is FPT when the out-branching and in-branching have the same root. By linking the two problems with the problem of whether a digraph has an out-branching with at least k leaves (a leaf is a vertex of out-degree zero), we first solve the problem of Bang-Jensen and Yeo (2008). We then develop a new digraph decomposition and using it prove that the problem of Bang-Jensen et al. (2016) is FPT for all digraphs.

1. Introduction

While both undirected and directed graphs are important in many applications, there are significantly more algorithmic and structural results for undirected graphs than for directed ones. The main reason is likely to be the fact that most problems on digraphs are harder than those on undirected graphs. The situation has begun to change: recently there appeared a number of important structural results on digraphs, see e.g. [16–18]. However, the progress was less pronounced with algorithmic results on digraphs, in particular, in the area of parameterized algorithms.

In this paper, we introduce a new decomposition for digraphs and show its usefulness by solving an open problem by Bang-Jensen, Saurabh and Simonsen [6]. We believe that our decomposition will prove to be helpful for obtaining further algorithmic and structural results on digraphs.

A digraph T is an *out-tree* (an *in-tree*) if T is an oriented tree with just one vertex s of in-degree zero (out-degree zero). The vertex s is the *root* of T. A vertex v of an out-tree (in-tree) is called a *leaf* if it has out-degree (in-degree) zero. If an out-tree (in-tree) T is a spanning subgraph of a digraph D, then T is an *out-branching* (an *in-branching*) of D. It is well-known that a digraph D contains an out-branching (in-branching) if and only if D has only one strongly connected component with no incoming (no outgoing) arc [3].

A well-known result in digraph algorithms, due to Edmonds, states that given a digraph D and a positive integer ℓ, we can decide whether D has ℓ arc-disjoint out-branchings in polynomial time [15]. The same result holds for ℓ arc-disjoint

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in-branchings. Inspired by this fact, it is natural to ask for a “mixture” of out- and in-branchings: given a digraph D and a pair u, v of (not necessarily distinct) vertices, decide whether D has an arc-disjoint out-branching T_u^+ rooted at u and in-branching T_v^- rooted at v. We will call this problem Arc-Disjoint Branchings.

Thomassen proved (see [2]) that the problem is NP-complete and remains NP-complete even if we add the condition that $u = v$. The same result still holds for digraphs in which the out-degree and in-degree of every vertex equals two [7]. The problem is polynomial-time solvable for tournaments [2] and for acyclic digraphs [8,10]. The single-root special case (i.e., when $u = v$) of the problem is polynomial time solvable for quasi-transitive digraphs[1] [4] and for locally semicomplete digraphs[2] [5].

An out-branching T^+ and an in-branching T^- are called *k-distinct* if $|A(T^+) \setminus A(T^-)| \geqslant k$. Bang-Jensen, Saurabh and Simonsen [6] considered the following parameterization of Arc-Disjoint Branchings.

> k-Distinct Branchings parametrised by k
> *Input:* A digraph D, an integer k.
> *Question:* Are there k-distinct out-branching T^+ and in-branching T^-?

They proved that k-Distinct Branchings is fixed-parameter tractable (FPT)[3] when D is strongly connected and conjectured that the same holds when D is an arbitrary digraph. Earlier, Bang-Jensen and Yeo [9] considered the version of k-Distinct Branchings where T^+ and T^- must have the same root and asked whether this version of k-Distinct Branchings, which we call Single-Root k-Distinct Branchings, is FPT.

The first key idea of this paper is to relate k-Distinct Branchings to the problem of deciding whether a digraph has an out-branching with at least k leaves via a simple lemma (see Lemma 1). The lemma and the following two results on out-branchings with at least k leaves allow us to solve the problem of Bang-Jensen and Yeo [9] and to provide a shorter proof for the above-mentioned result of Bang-Jensen, Saurabh and Simonsen [6] (see Theorem 3).

Theorem 1 *([1]). Let D be a strongly connected digraph. If D has no out-branching with at least k leaves, then the (undirected) pathwidth of D is bounded by* $O(k \log k)$.

Theorem 2 *([12,19]). We can decide whether a digraph D has an out-branching with at least k leaves in time*[4] $O^*(3.72^k)$.

The general case of k-Distinct Branchings seems to be much more complicated. We first introduce a version of k-Distinct Branchings called k-Rooted Distinct Branchings, where the roots s and t of T^+ and T^- are fixed, and add arc ts to D (provided the arc is not in D) to make D strongly connected. This introduces a complication: we may end up in a situation where D has an out-branching with many leaves, and thereby potentially unbounded pathwidth, but the root of the out-branching is not s. To deal with this situation, our goal will be to *reconfigure* the out-branching into an out-branching rooted at s. In order to reason about this process, we develop a new digraph decomposition we call the *rooted cut decomposition*. The cut decomposition of a digraph D rooted at a given vertex r consists of a tree $\hat{T}$ rooted at r whose nodes are some vertices of D and subsets of vertices of D called *diblocks* associated with the nodes of $\hat{T}$.

Our strategy is now as follows. If $\hat{T}$ is *shallow* (i.e., it has bounded height), then any out-branching with sufficiently many leaves can be turned into an out-branching rooted at s without losing too many of the leaves. On the other hand, if $\hat{T}$ contains a path from the root of $\hat{T}$ with sufficiently many non-degenerate diblocks (diblocks with at least three vertices), then we are able to show immediately that the instance is positive. The remaining and most difficult issue is to deal with digraphs with decomposition trees that contain long paths of diblocks with only two vertices, called *degenerate* diblocks. In this case, we employ two reduction rules which lead to decomposition trees of bounded height.

The paper is organized as follows. In the next section, we provide some terminology and notation on digraphs used in this paper. In Section 3, we prove Theorem 3. Section 4 is devoted to proving that Rooted k-Distinct Branchings is FPT for all digraphs using cut decomposition and Theorems 1 and 2. We conclude the paper in Section 5, where some open parameterized problems on digraphs are mentioned.

[1] A digraph $D = (V, A)$ is quasi-transitive if for every $xy, yz \in A$ there is at least one arc between x and z, i.e. either $xz \in A$ or $zx \in A$ or both.

[2] A digraph $D = (V, A)$ is locally semicomplete if for every $xy, xz \in A$ there is at least one arc between y and z and for every $yx, zx \in A$ there is at least one arc between y and z. Tournaments and directed cycles are locally semicomplete digraphs.

[3] Fixed-parameter tractability of k-Distinct Branchings means that the problem can be solved by an algorithm of runtime $O^*(f(k))$, where O^* omits not only constant factors, but also polynomial ones, and f is an arbitrary computable function. The books [11,13] are excellent recent introductions to parameterized algorithms and complexity.

[4] The algorithm of [19] runs in time $O^*(4^k)$ and its modification in [12] in time $O^*(3.72^k)$.

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