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Unit interval vertex deletion: Fewer vertices are relevant ☆

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ABSTRACT

The unit interval vertex deletion problem asks for a set of at most k vertices whose deletion from a graph makes it a unit interval graph. We develop an $O(k^4)$-vertex kernel for the problem, significantly improving the $O(k^{53})$-vertex kernel of Fomin et al. (2013) [11]. We start from a constant-approximation solution and study its interaction with other vertices, which induce a unit interval graph. We partition vertices of this unit interval graph into cliques in a naive way, and pick a small number of representatives from each of them. Our constructive proof for the correctness of our algorithm, using interval models, greatly simplifies the "destructive" proofs, based on destroying forbidden structures, for similar problems in literature. Our algorithm can be implemented in $O(mn+n^2)$ time, where n and m denote respectively the numbers of vertices and edges of the input graph.

1. Introduction

A graph is a *unit interval graph* if its vertices can be assigned to unit-length intervals on the real line such that there is an edge between two vertices if and only if their corresponding intervals intersect. Given a graph G and an integer k, the *unit interval vertex deletion* problem asks whether there is a set of at most k vertices whose deletion makes G a unit interval graph. According to Lewis and Yannakakis [17], this problem is NP-complete.

This paper approaches this problem by studying its kernelization. Given an instance (G, k) of unit interval vertex deletion, a *kernelization algorithm* produces in polynomial time an equivalent instance (G', k')—(G, k) is a yes-instance if and only if (G', k') is a yes-instance—such that $k' \leq k$ and the *kernel size* (the number of vertices in G') is upper bounded by some function of k'. Fomin et al. [11] presented an $O(k^{53})$-vertex kernel for the problem, which we improve to the following. As usual, n and m denote respectively the numbers of vertices and edges of the input graph.

Theorem 1.1. *The unit interval vertex deletion problem has an $O(k^4)$-vertex kernel, and it can be produced in $O(nm+n^2)$ time.*

The structures of unit interval graphs have been well studied and well understood. It is known that a graph is a unit interval graph if and only if it contains no claw, net, tent, (as depicted in Fig. 1), or any hole, i.e., an induced cycle on at least four vertices [20,21]. One can decide in linear time whether a given graph is a unit interval graph; if it is not, we can

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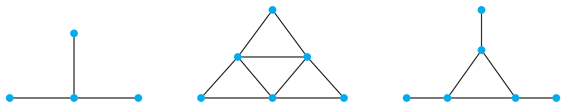

(a) claw (b) tent (c) net

Fig. 1. Forbidden induced graphs.

obtain a forbidden induced subgraph in the same time [12]. The unit interval vertex deletion problem can be equivalently defined as finding a set of at most k vertices that hits all forbidden induced subgraphs of the input graph.

The vertex deletion problem has been defined on many other graph classes and has been intensively studied. It is the easy case when the objective graph class has a finite set of forbidden induced subgraphs: As observed by Flum and Grohe [10], the sunflower lemma of Erdős and Rado [9] can be used to produce a polynomial kernel for the vertex deletion problem to this graph class. However, kernels produced as such tend to be very large. Furthermore, most interesting graph classes have an infinite number of forbidden induced subgraphs, hence the hard case. Another approach that works for both (and even for edge modification problems) is to start from a modulator, i.e., a set of vertices whose deletion leaves a graph in the objective graph class. With a modulator, we are allowed to use the properties of the objective graph class to analyze the rest of the graph. What is important is then the interaction between other vertices and the modulator [8,15,1].

Since holes of any length are forbidden in unit interval graphs, our problem belongs to the hard case. Hence both Fomin et al. [11] and the present paper use the modulator approach. Both of our modulators consist of two parts, first a set of vertices that hits all *small* forbidden induced subgraphs,—we use different thresholds for being small—and the second an optimal hitting set for long holes in the remaining graph. Recall that long holes behave very nicely in a graph free of small forbidden induced subgraphs: For example, a minimum hitting set for them can be found in linear time [4]. Thus, what differentiates the two algorithms is how to find the first part: Fomin et al. [11] used the sunflower lemma, while we use a constant-approximation algorithm [4].

We only need to proceed when the approximation algorithm produces a solution of $O(k)$ vertices. Thus, our modulator has a linear size, which is in a sharp contrast with the huge modulator of Fomin et al. [11] produced by the sunflower lemma. On the other hand, their modulator has an extra property that is not shared by ours. It guarantees that they only need to care about small forbidden induced subgraphs *inside the modulator*, thereby saving them a lot of trouble in the selection of relevant vertices. In our case, the interaction of the modulator with the rest of the graph is far more complicated, and the analysis is fundamentally different. In particular, the main technical difficulties present themselves at the analysis of the small forbidden induced subgraphs. This is exactly where our main technical ideas appear, which result in an algorithm and analysis significantly simpler than those of Fomin et al. [11].

Let (G, k) be the input instance and let M be the modulator. Our first idea is to *partition* the vertices of the unit interval subgraph $G - M$ into cliques and organize them in a linear way such that vertices in each clique are adjacent to only vertices in its two neighboring cliques. This is quite different from the widely used clique path decomposition, because in a clique path decomposition, (1) a vertex can appear in more than one clique; and (2) two vertices in cliques that are far away can be adjacent in the graph. We apply some simple reduction rules, after which we only need to proceed when there are $O(k^2)$ cliques in the partition of $G - M$. From each of these cliques at most $O(k^3)$ vertices are relevant. This implies a kernel of $O(k^5)$ vertices, and a more careful analysis leads to the smaller size claimed in Theorem 1.1.

Our second idea appears in the proof of the correctness of our algorithm. Our reduction rules are rather elementary and self-explanatory. The main step is to prove the irrelevance of the other vertices. We may assume that the reduction rules have been exhaustively applied to (G, k), and let G' denote the subgraph induced by the relevant vertices and the modulator. We need to show that if there is a solution V_- of size at most k to G', then it *must* also be a solution to G. Instead of showing the nonexistence of forbidden induced subgraph in $G - V_-$, (which would necessarily lead to an endless list of cases,) we build a unit interval model for $G - V_-$ out of a unit interval model for $G' - V_-$.

Jansen and Pilipczuk [15] recently studied the kernelization of the chordal vertex deletion problem and produced the first polynomial kernel. Their kernel has also a huge size, $O(k^{162})$ vertices, and was swiftly improved to $O(k^{26})$ by Agrawal et al. [1]. Both algorithms used approximation solutions as the modulators, for which they had to design a polynomial-time approximation algorithm.

Let us also mention the related parameterized algorithms (i.e., algorithms running in time $O(f(k) \cdot n^{O(1)})$ for some function f independent of n) for the problem, which have undergone a sequence of improvements. Recall that chordal graphs are those graphs containing no holes, and thus unit interval graphs are a subclass of chordal graphs. Toward a parameterized algorithm with $f(k) = \Omega(6^k)$, one can always dispose of all the claws, nets, and tents from the input graph, and then call the algorithm for the chordal vertex deletion problem [19,5] to break all holes in the remaining graph. Direct algorithms for unit interval vertex deletion were later reported in [2,13,4], and the current best algorithm runs in time $O(6^k \cdot (n + m))$. All the three direct algorithms use a two-phase approach. In [4], for example, the first phase breaks all claws, nets, tents, and C_4's, while the second phase deals with the remaining {claw, net, tent, C_4}-free graphs, on which the problem can be solved in polynomial time. A simple adaptation of this approach leads to an $O(nm + n^2)$-time 6-approximation algorithm.

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