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Categorical aspects of inducing closure operators on graphs by sets of walks

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ABSTRACT

We study closure operators on graphs which are induced by sets of walks of identical lengths in these graphs. It is shown that the induction gives rise to a Galois correspondence between the category of closure spaces and that of graphs with walk sets. We study the two isomorphic subcategories resulting from the correspondence, in particular, the one that is a full subcategory of the category of graphs with walk sets. As examples, we discuss closure operators that are induced by path sets on some natural graphs on the digital plane $\mathbb{Z}^2$. These closure operators are shown to include the well known Marcus–Wyse and Khalimsky topologies, thus indicating the possibility of using them as convenient background structures on the digital plane for the study of geometric and topological properties of digital images.

1. Introduction

As a basic field of discrete mathematics, graph theory finds a wide spectrum of applications. In particular, a special branch of discrete geometry devised for the study of geometric and topological properties of digital images, digital topology, is based on the use of topological aspects of graph theory such as graph connectedness. There are two approaches to digital topology, traditional [12,14] and topological [8,10,19]. While the former employs purely graph-theoretic tools, the latter is based on topological methods. Since both approaches have their specific advantages, it is most desirable to find bridges between them by studying the relationships between graph theory and topology. Such relationships were dealt with by several authors who investigated correspondences between directed graphs (i.e., sets with a binary relation) and topologies or closure operators, see e.g. [2,5,13,16]. The correspondences usually considered associate an Alexandroff topology (or completely additive closure operator) with graphs in a very natural way, thus obtaining the so-called left topology (left closure operator), in which the closure of a set A equals $A \cup \{x$; there is an edge (a, x) with $a \in A\}$, and, dually, the so-called right topology (right closure operator). However, up to now, only little effort has been exerted to investigate correspondences between simple graphs and spaces more general than the Alexandroff ones. The aim of this note is to proceed with such an investigation. We will focus on studying relationships between (simple) graphs and closure operators that generalize topologies (given by Kuratowski closure operators). It was shown in [17] that such closure operators provide richer scale of instruments for the needs of digital topology than the two topologies usually used, the Khalimsky [8] and Marcus–Wyse [13] ones.

The present paper is a continuation of the author's study of the topic started in [18] and [20]. We will deal with graphs each having a set specified of walks with identical lengths. Such graphs, with special walk sets called path partitions, were

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introduced and studied in [18] where their geometric properties were discussed based on a special concept of connectedness. The closure operators induced, in a certain way, on graphs by walk sets were discussed in [20] with some interesting relationships between the graphs and the induced closure operators shown. Building on [18], we will discover some more relationships between graphs with walk sets and the induced closure operators and, moreover, such relationships will be regarded in terms of category theory. More precisely, we will show that the induction gives rise to a Galois correspondence between the category of closure spaces and that of graphs with walk sets. The Galois correspondence will be studied and the results achieved will be demonstrated by examples of closure operators induced on (graphs on) the digital plane by certain walk sets.

2. Preliminaries

For the graph-theoretic terminology, we refer to [7]. By a *graph* $G = (V, E)$ we understand an undirected simple graph without loops with $V \neq \emptyset$ the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ the set of *edges*. We will say that G is a graph *on* V. As usual, two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. A key role will be played by the concept of a walk. Unlike the usual walks, in the present paper, the walks are allowed to be transfinite. More precisely, given an ordinal $\alpha > 1$, by an α-*walk* (briefly, a *walk*) in G we understand a sequence (of type α) $(x_i|\ i < \alpha)$ of vertices of V such that x_i is adjacent to x_{i+1} whenever $i + 1 < \alpha$. If $\alpha > 1$ is a finite ordinal, then $\alpha - 1$ is called the *length* of the walk $(x_i|\ i < \alpha)$. An α-walk is called an α-*path* (briefly, a *path*) if its members are pairwise different.

By a *closure operator* u on a set X, we mean a topology in Čech's sense [3], i.e., a map u: exp $X \to$ exp X (where exp X denotes the power set of X) that is

(i) grounded (i.e., $u\emptyset = \emptyset$),
(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).

The pair (X, u) is then called a *closure space*. Thus, the usual topologies (i.e., Kuratowski closure operators — cf. [6]) are the closure operators u on X that are

(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$).

Given closure spaces (X, u) and (Y, v), a map $f : X \to Y$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

As usual, we identify cardinals with initial ordinals (accepting so the Axiom of Choice). Given an ordinal α, we denote by $\langle\alpha\rangle$ the least cardinal n with $\alpha \le n$.

Let $m > 1$ be a cardinal. A closure operator u on a set X and the closure space (X, u) are called an S_m-*closure operator* and an S_m-*closure space* (briefly, an S_m-*space*), respectively, if the following condition is satisfied:

$A \subseteq X \Rightarrow uA = \bigcup\{uB;\ B \subseteq A, \text{card } B < m\}$.

In [4], S_2-closure operators and S_2-spaces are called *quasi-discrete*. S_2-topologies (S_2-topological spaces) are called *Alexandroff topologies* (*Alexandroff spaces*) — cf. [8]. Clearly, every S_2-closure operator is additive and, if $m \le \aleph_0$, then every additive S_m-closure operator is an S_2-closure operator. Since every S_m-closure operator is an S_n-closure operator whenever $m \le n$, it is useful to know, for a given closure operator u on X, the minimal cardinal m for which u is an S_m-closure operator. Such a minimal cardinal is an important invariant of the closure space (X, u) as mentioned in [3].

We will use some basic topological concepts (see e.g. [6]) naturally extended from topological to closure spaces. In particular, given a closure space (X, u), a subset $A \subseteq X$ is said to be *closed* if $uA = A$ (and it is said to be *open* if its complement in X is closed). If u, v are closure operators on a set X, then we put $u \le v$ if $uA \subseteq vA$ for every subset $A \subseteq X$. Clearly, $\le$ is a partial order on the set of all closure operators on X. If $u \le v$, then u is said to be *finer* than v and v is said to be *coarser* than u. Note that, for topologies given by open sets, just the converse partial order is usually used.

For the categorical terminology used see [1]. All categories are considered to be constructs, i.e., concrete categories over *Set* (the category of sets and maps), and all functors are assumed to be concrete, i.e., to preserve the underlying sets and to be identities for morphisms (so that the functors are given by determining them as maps on objects). Recall that, given a pair of objects $A = (X, \rho)$ and $B = (X, \sigma)$ of a category, we write $\rho \le \sigma$ if $\mathrm{id}_X : (X, \rho) \to (X, \sigma)$ is a morphism. We will also write $A \le B$ in this case. Given categories $\mathcal{X}$, $\mathcal{Y}$ and functors $F, G : \mathcal{X} \to \mathcal{Y}$, we write $F \le G$ if $F(A) \le G(A)$ for every object $A \in \mathcal{X}$. A Galois correspondence between categories $\mathcal{X}$ and $\mathcal{Y}$ is a pair of functors (L, R), $L : \mathcal{Y} \to \mathcal{X}$ and $R : \mathcal{X} \to \mathcal{Y}$, such that $L \circ R \le \mathrm{id}_{\mathcal{X}}$ and $R \circ L \ge \mathrm{id}_{\mathcal{Y}}$.

3. A categorical Galois correspondence that arises from inducing closure operators on graphs by walk sets

Given a graph G and an ordinal $\alpha > 1$, we denote by $\mathcal{W}_\alpha(G)$ the set of all α-walks in G. Every subset $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ will be called an α-*walk set* or, briefly, a *walk set* in G. If every element of $\mathcal{B}$ is even a path, then $\mathcal{B}$ will be called an α-*path set* or, briefly, a *path set* in G.

Let X be a set, $\alpha > 1$ an ordinal, and $\mathcal{B} \subseteq X^\alpha$ (where X^α denotes the set of all sequences of type α with all members from X) a subset such that $(x_i|\ i < \alpha) \in \mathcal{B}$ implies $x_i \neq x_{i+1}$ whenever $0 < i + 1 < \alpha$. Put $E_{\mathcal{B}} = \{\{x_i, x_{i+1}\};\ 0 < i + 1 < \alpha\}$.