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A note on maximum differential coloring of planar graphs

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ABSTRACT

We study the *maximum differential coloring problem*, where the vertices of an n-vertex graph must be labeled with distinct numbers ranging from 1 to n, so that the minimum absolute difference between two labels of any two adjacent vertices is maximized. As the problem is NP-hard for general graphs [16], we consider planar graphs and subclasses thereof. We prove that the maximum differential coloring problem remains NP-hard, even for planar graphs. We also present tight bounds for regular caterpillars and spider graphs. Using these new bounds, we prove that the Miller–Pritikin labeling scheme [19] for forests is optimal for regular caterpillars and for spider graphs.

1. Introduction

The four color theorem states that only four colors are needed to color any map, so that no neighboring countries share the same color. However, if the countries in the map are not all contiguous, then the result no longer holds [6]. In order to avoid ambiguity, this necessitates the use of a unique color for each country. As a result, the number of colors needed is equal to the number of countries.

Given a map, define the country graph $G = (V, E)$ to be the graph where countries are vertices and two countries are connected by an edge if they share a nontrivial border. In the *maximum differential coloring problem* [16] the goal is to find a labeling of the n vertices of graph G with distinct numbers ranging from 1 to n (treated as *colors*), which maximizes the absolute label difference among adjacent vertices. More formally, let $C = \{c \mid c : V \rightarrow \{1, 2, \ldots, |V|\}\}$ be the set of one-to-one functions for labeling the vertices of G. For any $c \in C$, the *differential coloring* achieved by c is $\min_{(i,j)\in E} |c(i) - c(j)|$. We seek the labeling function $c \in C$ that achieves the maximum differential coloring: $\mathrm{DC}(G) = \max_{c\in C} \min_{(i,j)\in E} |c(i) - c(j)|$, which is the *differential chromatic number* of G. We also say that G admits a *differential coloring of value k*, if there is some labeling function $c \in C$ so that $\min_{(i,j)\in E} |c(i) - c(j)| = k$.

The maximum differential coloring problem is in a sense the opposite of the well-studied *bandwidth minimization problem*, which is known to be NP-complete [20,21]. Optimal algorithms for the bandwidth minimization problem are known only for restricted classes of graphs, e.g., caterpillars with hair length 1 [18], caterpillars with hair length 3 [1], chain graphs [14], co-graphs [28], bipartite permutation graphs [11], AT-free graphs [8]. As in many graph theoretic maximization vs minimization problems (e.g., shortest vs longest path), results for bandwidth minimization do not translate into results for maximum differential coloring. Although the maximum differential coloring problem is less known than the bandwidth minimization problem, it has received considerable attention recently. In addition to map-coloring [6], the problem is motivated by the

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radio frequency assignment problem, where n transmitters have to be assigned n frequencies, so that interfering transmitters have frequencies as far apart as possible [10].

1.1. Previous work

The maximum differential coloring problem was initially studied in the context of multiprocessor scheduling under the name "separation number" by Leung et al. [16], who showed that the problem is NP-complete. Twenty years later, Yixun et al. [29] studied the same problem under the name "dual-bandwidth" and gave several upper bounds, including the following simple bound for connected graphs:

DC-Property 1. For any connected graph G, $\mathrm{DC}(G) \leq \lfloor \frac{n}{2} \rfloor$ [29].

The proof is straightforward: one of the vertices of G has to be labeled $\lceil \frac{n}{2} \rceil$ and since G is connected that vertex must have at least one neighbor which (regardless of its label) would make the difference along that edge at most $\lfloor \frac{n}{2} \rfloor$.

The maximum differential coloring problem is also known as the "anti-bandwidth problem" [3]. Heuristics for the maximum differential coloring problem have been suggested by Duarte et al. [5] using LP-formulation, by Bansal et al. [2] using memetic algorithms and by Hu et al. [12] using spectral based methods. Another line of research focuses on solving the maximum differential coloring problem optimally for special classes of graphs, e.g., Hamming graphs [4], meshes [25], hypercubes [23,26], complete binary trees [27] and complete k-ary trees for odd values of k [3]. Isaak et al. [13] give a greedy algorithm for the differential chromatic number of complement of interval and threshold graphs by computing the k-th power of a Hamiltonian path. Weili et al. [27] compute the differential chromatic number for what they call "caterpillars" (but which differ from the standard graph-theoretic caterpillars).

Miller and Pritikin [19] describe a labeling scheme which, for a forest G with bipartition U and V, gives a differential coloring value equal to the size of the smaller vertex set, i.e., $\min\{|U|, |V|\}$. This approach can be summarized as follows. Say, without loss of generality, that $|U| \leq |V|$. The vertices in U are labeled with labels from the "minority interval" $I_{\min} = [1, |U|]$, while the vertices in V are labeled with labels from the "majority interval" $I_{\text{maj}} = [|U|+1, |V|]$. Since the average degree of the vertices in V is $(|U| + |V| - 1)/|V| < 2$, there exists at least one vertex in V, say v, with degree ≤ 1. Based on the vertex v, a vertex $u \in U$ is chosen as follows: If $deg(v) = 1$, then u is the neighbor of v. Otherwise, u is arbitrarily chosen from U. Both v and u are then labeled with the smallest available labels from I_{maj} and $I_{\min}$, respectively, and removed from G. This procedure is repeated until U is empty. The remaining vertices in V (if any) are labeled with the remaining available labels in I_{maj}. Note that when a vertex $u \in U$ is labeled, a vertex $v \in V$ is also labeled. Hence, as long as U is non-empty, the number of labeled vertices of U is equal to the number of labeled vertices of V. This implies that the minimum label difference between any two neighboring vertices in G is at least $|U|$.

Equitable coloring [9] is also related to differential coloring. Formally, an equitable coloring is an assignment of colors to the vertices of a graph, so that no two adjacent vertices have the same color and the number of vertices in any two color classes differ by at most one. The problem of deciding whether a graph admits an equitable coloring with no more than three colors is NP-complete [15]. If a graph G has differential chromatic number k, then the vertices labeled $[1, k], [k+1, 2k] \cdots$ form equitably colored classes and so G has an equitable coloring with $\lfloor \frac{n}{k} \rfloor + 1$ colors. Lin et al. [17] describe a (sub-optimal) labeling for connected bipartite graphs with a differential coloring of value $\lfloor \frac{n}{\Delta} \rfloor$, where Δ is the max degree, using the relationship between the anti-bandwidth problem and the equitable coloring problem.

Another related problem is the *channel assignment problem* [24], in which each edge has a weight and the objective is to find a labeling of the vertices, so that the difference between the labels of the endpoints of each edge is at least equal to its weight. However, the same label can be used by multiple vertices.

1.2. Preliminaries

Let $G = (V, E)$ be an undirected graph. We denote the number of vertices of G by n, i.e., $n = |V|$. The degree of vertex $u \in V$ is denoted by $d(u)$. The degree of graph G is then defined as: $\Delta(G) = \max_{u \in V} d(u)$.

A *caterpillar* is a tree in which removing all leaves results in a path; see Fig. 1a. Thus, a caterpillar consists of a simple path, called the "spine", and each spine vertex is adjacent to a certain number of leaves, called the "legs". In our algorithms we assume that spine vertices are indexed in the order they appear along the spine. Hence, an *odd-indexed* (*even-indexed*) *spine vertex* is one that appears in an odd-indexed (even-indexed) position along the spine. In caterpillars, Δ refers to the maximum number of legs of any spine vertex. In a *regular caterpillar*, every spine vertex has the same number of legs. A *spider* is a graph with a center vertex connected to several disjoint paths; see Fig. 1b. The vertices of a spider have *levels*, according to their distance from the center. In a spider, N_e, N_o and N_l denote the number of even level, number of odd level and number of vertices in level l, respectively. A *radius-k star* graph is a spider with all paths of the same length k; see Fig. 1c.