Journal of Discrete Algorithms ••• (••••) •••–•••

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Journal of Discrete Algorithms

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# Greedy algorithms and poset matroids ☆

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ARTICLE INFO

*Article history:*
Received 3 August 2013
Received in revised form 11 February 2014
Accepted 10 July 2014
Available online xxxx

*Keywords:*
Greedy algorithm
Matroid
Poset
Abstract simplicial complex

ABSTRACT

We generalize the matroid-theoretic approach to greedy algorithms to the setting of poset matroids, in the sense of Barnabei, Nicoletti and Pezzoli (1998) [1]. We illustrate our result by providing a generalization of Kruskal algorithm (which finds a minimum spanning subtree of a weighted graph) to abstract simplicial complexes.



## 1. Introduction

An *independence system* is a pair $(E, \mathcal{F})$ such that $E$ is a finite set and $\mathcal{F}$ is a down-set of the Boolean algebra $\wp(E)$ (i.e. a subset of $\wp(E)$ such that, if $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$). A *matroid* is an independence system satisfying the following axiom: for any $A, B \in \mathcal{F}$ such that $|B| = |A| + 1$, there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$.

In the paper [1] the authors propose a generalization of the notion of matroid where the ground set is equipped with a partial order. The central definition of their work is the following: a *poset matroid* is a pair $(P, \mathcal{I})$ where $P$ is a finite partially ordered set and $\mathcal{I}$ is a nonempty family of up-sets of $P$ satisfying the following properties:

(i) if $X, Y$ are up-sets of $P$ such that $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$;
(ii) for every $X, Y \in \mathcal{I}$ with $|X| < |Y|$, there exists $y \in Max(Y \setminus X)$ such that $X \cup \{y\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the *independent sets* of the poset matroid. To understand the above definition, we recall that an up-set (resp., down-set) of a poset $P$ is a subset $S$ of $P$ such that, if $x \in S$ and $x \le y$ (resp. $x \ge y$), then $y \in S$. Moreover, for any $S \subseteq P$, we denote by $Max(S)$ the set of maximal elements of $S$.

We remark that the definition given here differs from the original one in [1], which is given in terms of the notion of *basis*. However, the two definitions are clearly equivalent, as is shown in [1].

Given this generalized notion of matroid, it is natural to try to generalize notions and results of matroid theory to the context of poset matroids. Among the open problems proposed in [1], the last one is the following: is it possible to generalize the generic greedy algorithm to the setting of poset matroids? To better understand this problem, recall that there is a strong relationship between greedy algorithms and the notion of matroid, which we will briefly summarize below.

☆ Partially supported by MIUR PRIN 2010–2011 grant "Automi e Linguaggi Formali: Aspetti Matematici e Applicativi", code H41J12000190001.
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http://dx.doi.org/10.1016/j.jda.2014.07.005
1570-8667/

Given a *weight function* $w : E \to \mathbf{R}^+$, we consider the following problem:

**input**: an independence system $(E, \mathcal{F})$ and a weight function $w : E \to \mathbf{R}^+$.
**output**: a set $M \in \mathcal{F}$ such that $w(M) = \sum_{x \in M} w(x)$ is maximum.

The *greedy algorithm* for the independence system $(E, \mathcal{F})$ attempts to solve the above problem, and consists of the following procedure:

**Algorithm 1:** GREEDY($(E, \mathcal{F})$, $w$).

```
S := ∅;
Q := E;
while Q ≠ ∅ do
    find m ∈ Q having maximum weight;
    Q := Q \ {m};
    if S ∪ {m} ∈ F then
        S := S ∪ {m};
return S;
```

The procedure GREEDY tries to find a global solution by making the local best choice at each step. Unfortunately, such an algorithm is not always correct (that is, it does not solve the above problem in general). The following theorem by Edmonds and Rado [3,10] tells us in which cases it works.

**Theorem 1.1.** *Given an independence system $(E, \mathcal{F})$, the following statements are equivalent:*

a) *for any weight function $w$, GREEDY is correct on input $(E, \mathcal{F})$, $w$;*
b) *$(E, \mathcal{F})$ is a matroid.*

In the next section we will generalize the Edmonds–Rado theorem to the setting of poset matroid. In Section 3 we will see how our generalization can be used to find an analog of Kruskal algorithm, which determines a minimum spanning subtree of a weighted graph, where the graph is replaced by an abstract simplicial complex. Finally, in the last section we will give some hints to relate our work with previous approaches on the same (or perhaps similar) subject. In particular, we will try to compare our results with similar ones obtained, for instance, by Faigle and Fujishige. Since the detailed discussion of analogies and differences between these approaches clearly requires the knowledge of the theory we are going to develop in the present paper, we have preferred to postpone such a discussion at the end of the article. In this way the reader should have more information in order to fully appreciate all the details of the comparison.

In closing this introduction we briefly recall a standard notion concerning posets that will be useful throughout the paper. Given two posets $P, Q$, a function $f : P \to Q$ is said to be *order-preserving* (resp., *order-reversing*) whenever, for all $x, y \in P$, $x \leq y$ implies that $f(x) \leq f(y)$ (resp., $f(x) \geq f(y)$).

## 2. The Edmonds–Rado theorem for poset matroids

Given a poset $P$, let $\mathcal{I}$ be a family of up-sets of $P$ satisfying condition (i) in the definition of poset matroid (i.e. $\mathcal{I}$ is a down-set of up-sets of $P$). Call such a pair $(P, \mathcal{I})$ a *po-independence system*.

Consider the following problem:

**input**: a po-independence system $(P, \mathcal{I})$ and an order-preserving weight function $w : P \to \mathbf{R}^+$.
**output**: an up-set $M \in \mathcal{I}$ such that $w(M) = \sum_{x \in M} w(x)$ is maximum.

To solve it we try to adapt the greedy algorithm as follows:

**Algorithm 2:** PGREEDY($(P, \mathcal{I})$, $w$).

```
S := ∅;
Q := P;
while Q ≠ ∅ do
    find a maximal element m ∈ Q having maximum weight;
    Q := Q \ {m};
    if S ∪ {m} ∈ I then
        S := S ∪ {m};
return S;
```