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Complete algebraic solution of multidimensional optimization problems in tropical semifield

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A B S T R A C T

We consider multidimensional optimization problems that are formulated in the framework of tropical mathematics to minimize functions defined on vectors over a tropical semifield (a semiring with idempotent addition and invertible multiplication). The functions, given by a matrix and calculated through multiplicative conjugate transposition, are nonlinear in the tropical mathematics sense. We start with known results on the solution of the problems with irreducible matrices. To solve the problems in the case of arbitrary (reducible) matrices, we first derive the minimum value of the objective function, and find a set of solutions. We show that all solutions of the problem satisfy a system of vector inequalities, and then use these inequalities to establish characteristic properties of the solution set. Furthermore, all solutions of the problem are represented as a family of subsets, each defined by a matrix that is obtained by using a matrix sparsification technique. We describe a backtracking procedure that allows one to reduce the brute-force generation of sparsified matrices by skipping those, which cannot provide solutions, and thus offers an economical way to obtain all subsets in the family. Finally, the characteristic properties of the solution set are used to provide complete solutions in a closed form. We illustrate the results obtained with simple numerical examples.

1. Introduction

Tropical (idempotent) mathematics, which deals with the theory and applications of semirings with idempotent addition, dates back to a few seminal works [26,3,7,11,30,27] which appeared in the early 1960s. Today, tropical mathematics is a rapidly evolving area (see, e.g., recent publications [14,8,10,13,9,2,25,24]), which offers a useful analytical and computational framework to solve many recent problems in operations research, computer science and other fields. These problems can be formulated and solved as optimization problems in the tropical mathematics setting, hence are referred to as tropical optimization problems. Typical examples of the application areas of tropical optimization include project scheduling [31,28,1,16,17,21], location analysis [12,29,15], and decision making [4–6,19].

Many tropical optimization problems are formulated to minimize or maximize functions defined on vectors over idempotent semifields (semirings with multiplicative inverses). These problems may have functions to optimize (objective functions), which can be linear or non-linear in the tropical mathematics sense, and constraints, which can take the form of vector inequalities and equalities. Some problems have direct, explicit solutions obtained using general assumptions. For other problems, only algorithmic solutions under restrictive conditions are known, which apply iterative numerical pro-

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cedures to find a solution if it exists, or to indicate infeasibility of the problem otherwise. A short overview of tropical optimization problems and their solutions can be found in [17].

In this paper, we consider the tropical optimization problems as to

$$\text{minimize} \quad (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{x}, \qquad \text{minimize} \quad \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{x},$$

where $\boldsymbol{A}$ is a given square matrix, $\boldsymbol{x}$ is an unknown vector, and the minus sign in the superscript serves to specify conjugate transposition of vectors. Since the objective function of the first problem is involved as a component of the composite objective function of the second, these problems are referred below to as component and composite problems for short.

As applications, variants of these problems occur, for instance, in optimal project scheduling under the minimum flow time criterion [16,17,21], and in multicriteria decision making with pairwise comparisons [19].

Partial solutions of the problems were obtained in [22], which specify a substantial part, but not all, of the solution sets for both irreducible and reducible matrices. The main purpose of this paper is to continue the investigation to derive complete solutions describing the entire solution set for general problems with arbitrary matrices. We follow the approach developed in [18] and based on a characterization of the solution set. We show that all solutions of the problems satisfy a vector inequality or a system of inequalities, and subsequently use these inequalities to establish characteristic properties of the solution set. The solutions are represented as a family of solution subsets, each defined by a matrix that is obtained by using a matrix sparsification technique.

Furthermore, we describe a backtracking procedure that allows one to reduce the brute-force generation of the matrices by skipping those, which cannot provide solutions, and thus offers an economical way to obtain all subsets in the family. Finally, the characteristic properties of the solution set are applied to provide a complete solution in a closed form. The results obtained are illustrated with illuminating numerical examples.

This paper further extends and supplements the results presented in the conference paper [20], which examined only the problem with component objective function. The current paper further improves the presentation of the solution of the component problem, and offers new results on the solution of the composite problem.

The rest of the paper is organized as follows. In Section 2, we give a brief overview of basic definitions and preliminary results of tropical algebra. Section 3 formulates the tropical optimization problems under study, and presents known solutions. In Section 4, we investigate the first problem with a component objective function. As a result, a complete solution of the problem with reducible matrix is obtained in a compact vector form. The results obtained are then extended to the solution of the problem with composite objective function in Section 5. Finally, Section 6 offers concluding remarks and suggestions for further research.

2. Preliminary definitions and results

We start with a brief overview of the preliminary definitions and results of tropical algebra to provide an appropriate formal background for the development of solutions for the tropical optimization problems in the subsequent sections. The overview is mainly based on the results in [23,22,16,17,21], which offer a useful framework to obtain solutions in a compact vector form, ready for further analysis and practical implementation. Additional details on tropical mathematics at both introductory and advanced levels can be found in many recent publications, including [14,8,10,13,9,2,25,24].

2.1. Idempotent semifield

An *idempotent semifield* is a system $(\mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes)$, where $\mathbb{X}$ is a nonempty set endowed with associative and commutative operations, addition $\oplus$ and multiplication $\otimes$, which have as neutral elements the zero $\mathbb{0}$ and the one $\mathbb{1}$. Addition is idempotent, that is $x \oplus x = x$ for all $x \in \mathbb{X}$. Multiplication distributes over addition, has $\mathbb{0}$ as absorbing element, and is invertible, which gives any nonzero x its inverse x^{-1} such that $x \otimes x^{-1} = \mathbb{1}$.

Idempotent addition induces on $\mathbb{X}$ a partial order such that $x \leq y$ if and only if $x \oplus y = y$. With respect to this order, both addition and multiplication are monotone, which means that, for all $x, y, z \in \mathbb{X}$, the inequality $x \leq y$ entails that $x \oplus z \leq y \oplus z$ and $x \otimes z \leq y \otimes z$. Furthermore, inversion is antitone to take the inequality $x \leq y$ into $x^{-1} \geq y^{-1}$ for all nonzero x and y. The inequality $x \oplus y \leq z$ is equivalent to the pair of inequalities $x \leq z$ and $y \leq z$. Finally, since $x \oplus \mathbb{0} = x$ implies that $x \geq \mathbb{0}$ for any x, the zero $\mathbb{0}$ is the least element of $\mathbb{X}$. The partial order is assumed to extend to a total order on the semifield.

The power notation with integer exponents is routinely defined to represent iterated products for all $x \neq \mathbb{0}$ and integer $p \geq 1$ in the form $x^0 = \mathbb{1}$, $x^p = x \otimes x^{p-1}$, $x^{-p} = (x^{-1})^p$, and $\mathbb{0}^p = \mathbb{0}$. Moreover, the equation $x^p = a$ is assumed to have a unique solution $x = a^{1/p}$ for any a, which extends the notation to rational exponents. In what follows, the multiplication sign $\otimes$ is, as usual, dropped to save writing.

A typical example of a semifield is the system $(\mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, +)$, which is usually referred to as the *max-plus algebra*. In this semifield, the addition $\oplus$ is defined as max, and the multiplication $\otimes$ is as arithmetic addition $+$. The number $-\infty$ is taken as the zero $\mathbb{0}$, and 0 is as the one $\mathbb{1}$. For each $x \in \mathbb{R}$, the inverse x^{-1} coincides with the conventional opposite number $-x$. For any rational y, the power x^y corresponds to the arithmetic product $x \times y$. The order induced by idempotent addition complies with the natural linear order on $\mathbb{R}$.